

Coalgebraic Simulation.

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Abstract. Hello simulation!

1 Coalgebraic Simulation

We show the category of preorders with monotone functions between them with **PreOrd**. In the diagrams, any arrow that shows a functor, but does not have a label is showing a forgetful functor. Also, we use **Rel** to refer to the category of binary relations. Assuming $R \in \mathbf{Obj}(\mathbf{Rel})$ and $R \subseteq X_1 \times X_2$, and $S \in \mathbf{Obj}(\mathbf{Rel})$ and $S \subseteq Y_1 \times Y_2$, then a morphism $f: R \rightarrow S$ in this category is the pair (f_1, f_2) of morphisms in **Set**, where, $f_1: X_1 \rightarrow Y_1$ and $f_2: X_2 \rightarrow Y_2$, and for each $(x_1, x_2) \in R$ we have $(f_1(x_1), f_2(x_2)) \in S$. Also, we show projections of $R \in \mathbf{Obj}(\mathbf{Rel})$ with p_1 and p_2 that are morphisms in **Set**.

Definition 1.1 (A Preorder Over a Functor). Assuming $F: \mathbf{Set} \rightarrow \mathbf{Set}$ is a functor, we call $\sqsubseteq: \mathbf{Set} \rightarrow \mathbf{PreOrd}$ an order over the functor F iff the following diagram commutes:

$$\begin{array}{ccc} & & \mathbf{PreOrd} \\ & \nearrow \sqsubseteq & \downarrow \\ \mathbf{Set} & \xrightarrow{F} & \mathbf{Set} \end{array}$$

Definition 1.2 (Relation Lifting). Assuming $F: \mathbf{Set} \rightarrow \mathbf{Set}$ is a functor, then we call $\mathbf{Rel}(F): \mathbf{Rel} \rightarrow \mathbf{Rel}$ a relation lifting of F , where the following diagram commutes:

$$\begin{array}{ccc} \mathbf{Rel} & \xrightarrow{\mathbf{Rel}(F)} & \mathbf{Rel} \\ \downarrow & & \downarrow \\ \mathbf{Set} \times \mathbf{Set} & \xrightarrow{F \times F} & \mathbf{Set} \times \mathbf{Set} \end{array}$$

We take $\mathbf{Rel}(F): \mathbf{Rel} \rightarrow \mathbf{Rel}$ to be the functor that for an arbitrary functor F takes a relation R , where $R \in \mathbf{Obj}(\mathbf{Rel})$ and $R \subseteq X_1 \times X_2$, and gives the relation that is the image of the function $\langle Fp_1, Fp_2 \rangle: FR \rightarrow FX \times FY$.

Definition 1.3 (Bisimulation). For a functor $F: \mathbf{Set} \rightarrow \mathbf{Set}$, a bisimulation is a $\mathbf{Rel}(F)$ -coalgebra in **Rel**.

27 **Proposition 1.4.** *Assuming that (R, α) is a $\mathbf{Rel}(F)$ -coalgebra, where $\alpha = \beta_1 \times$
 28 β_2 in $\mathbf{Set} \times \mathbf{Set}$, then the following diagram commutes, and vice-versa:*

$$\begin{array}{ccccc} X_1 & \xleftarrow{p_1} & R & \xrightarrow{p_2} & X_2 \\ \beta_1 \downarrow & & \downarrow \beta & & \downarrow \beta_2 \\ FX_1 & \xleftarrow{Fp_1} & FR & \xrightarrow{Fp_2} & FX_2 \end{array}$$

29 We gave an introduction to Hughes and Jacobs paper. They also have a way
 30 to represent simulation relations. In the following, we try to find a suitable
 31 formalization for simulation relations, inspired by Hughes and Jacobs.

32 1.1 Relations as ~~Pullbacks~~ Spans(?)

33 We can not show every relation by pullbacks, but we can just show relations of
 34 the form

$$\{(a, b) \mid f(a) = g(b)\}$$

35 for some functions f and g , when we are in \mathbf{Set} , so we can not show every ob-
 36 ject in \mathbf{Rel} using this approach, including $\mathbf{Rel}_{\sqsubseteq}(F)(R) \sqsubseteq_{X_2}; \mathbf{Rel}(F)(R); \sqsubseteq_{X_1}$
 37 that is the target of simulation. Although we can show $\mathbf{Rel}_{\sqsubseteq}(F)(R) \sqsubseteq_{X_2}$
 38 $; \mathbf{Rel}(F)(R); \sqsubseteq_{X_1}$ as a span.

39 Assuming, we have a category \mathbf{C} , an object of the category of spans over \mathbf{C}
 40 is (R, X_1, X_2, p_1, p_2) in the form of the following diagram:

$$\begin{array}{ccc} & R & \\ p_1 \swarrow & & \searrow p_2 \\ X_1 & & X_2 \end{array}$$

41 A morphism from a span (R, X_1, X_2, p_1, p_2) to a span (S, Y_1, Y_2, q_1, q_2) is a mor-
 42 phism $f: R \rightarrow S$ in \mathbf{C} , for which exist $f_1: X_1 \rightarrow Y_i$ and $f_2: X_2 \rightarrow Y_j$, where
 43 $i, j \in \{1, 2\}$ and $i \neq j$, and they are in \mathbf{C} , that take part in the following com-
 44 muting diagram:

$$\begin{array}{ccccc} X_2 & \xleftarrow{p_2} & R & \xrightarrow{p_1} & X_1 \\ f_2 \downarrow & & \downarrow f & & \downarrow f_1 \\ Y_j & \xleftarrow{q_j} & S & \xrightarrow{q_i} & Y_i \end{array}$$

45 We define a F -simulation as the coalgebra of the object
 46 $\sqsubseteq_{X_1}; \mathbf{Rel}(F)(R); \sqsubseteq_{X_2}$ that has the following structure in \mathbf{C} :

$$\begin{array}{ccccc}
 \sqsubseteq_{X_1}; \mathbf{Rel}(F)(R); \sqsubseteq_{X_2} & \xrightarrow{\pi_1} & \sqsubseteq_{X_1}; \mathbf{Rel}(F)(R) & \xrightarrow{\varphi_1} & \sqsubseteq_{X_1} \xrightarrow{i_1^1} FX_1 \\
 \downarrow \pi_2 & \lrcorner & \downarrow \varphi_2 & \lrcorner & \downarrow i_2^1 \\
 & & FR & \xrightarrow{Fp_1} & FX_1 \\
 & & \downarrow Fp_2 & & \\
 \sqsubseteq_{X_2} & \xrightarrow{i_1^2} & FX_2 & & \\
 \downarrow i_2^2 & & & & \\
 FX_2 & & & &
 \end{array}$$

47

48 We show that if we consider a relation R and its opposite are both simula-
 49 tion relations, then R is a bisimulation. To reach to that goal, we give a formal
 50 definition of what we mean by the opposite of R in our categorical setting that
 51 we show with R^{op} . $(R^{\text{op}}, p'_1, p'_2)$ is a span, that is isomorphic to R via morphism
 52 $s: R \rightarrow R^{\text{op}}$ in \mathbf{Rel} that we call swap, and it commutes in the following commu-
 53 tative diagram:

$$\begin{array}{ccccc}
 X_2 & \xleftarrow{p_2} & R & \xrightarrow{p_1} & X_1 \\
 \text{id} \downarrow & & \downarrow s & & \downarrow \text{id} \\
 X_2 & \xleftarrow{p'_1} & R^{\text{op}} & \xrightarrow{p'_2} & X_1
 \end{array}$$

54

55 **Lemma 1.5.** *The relation $(\sqsubseteq_{X_1}; \mathbf{Rel}(F)(R); \sqsubseteq_{X_2})^{\text{op}}$ is isomorphic to $\sqsubseteq_{X_2^{\text{op}}}$
 56 $; \mathbf{Rel}(F)(R^{\text{op}}); \sqsubseteq_{X_1^{\text{op}}}$.*

57 *Proof.* We set $s_1: \sqsubseteq_{X_1} \rightarrow \sqsubseteq_{X_1}^{\text{op}}$ and $s_2: \sqsubseteq_{X_2} \rightarrow \sqsubseteq_{X_2}^{\text{op}}$ to be the swaps of \sqsubseteq_{X_1} and
 58 \sqsubseteq_{X_2} , respectively. Since we have

$$\begin{aligned}
 i_1'^1 \cdot s_1 \cdot \varphi_1 &= i_2^1 \cdot \varphi_2 \\
 &= Fp_1 \cdot \varphi_2 \\
 &= Fp_2' \cdot Fs \cdot \varphi_2,
 \end{aligned}$$

59 there exists the morphism $s'': \sqsubseteq_{X_1}; \mathbf{Rel}(F)(R) \rightarrow \mathbf{Rel}(F)(R^{\text{op}}); \sqsubseteq_{X_1}^{\text{op}}$ depicted
 60 in the following commutative diagram:

$$\begin{array}{ccccc}
 \sqsubseteq_{X_1}; \mathbf{Rel}(F)(R) & \xrightarrow{\varphi_1} & \sqsubseteq_{X_1} & & \\
 \downarrow \varphi_2 & \searrow s'' & \downarrow s_1 & & \\
 FR & & \mathbf{Rel}(F)(R^{\text{op}}); \sqsubseteq_{X_1}^{\text{op}} & \xrightarrow{\varphi_2} & \sqsubseteq_{X_1}^{\text{op}} \\
 \downarrow Fs & & \downarrow \varphi_1' & \lrcorner & \downarrow i_1'^1 \\
 & & FR^{\text{op}} & \xrightarrow{Fp_2'} & FX_1
 \end{array}$$

61 Similarly, we get $s''^{-1}: \sqsubseteq_{X_1}; \mathbf{Rel}(F)(R) \rightarrow \mathbf{Rel}(F)(R^{\text{op}}); \sqsubseteq_{X_1}^{\text{op}}$ since

$$\begin{aligned} i_2^1 \cdot s_1^{-1} \cdot \varphi'_2 &= i_1^1 \cdot \varphi'_2 \\ &= Fp'_2 \cdot \varphi'_1 \\ &= Fp_1 \cdot Fs_1^{-1} \cdot \varphi'_1, \end{aligned}$$

62 and it is depicted in the following diagram:

$$\begin{array}{ccccc} \mathbf{Rel}(F)(R^{\text{op}}); \sqsubseteq_{X_1}^{\text{op}} & \xrightarrow{\varphi'_2} & \sqsubseteq_{X_1} & & \\ \varphi'_1 \downarrow & \searrow s''^{-1} & \searrow s_1^{-1} & & \\ FR^{\text{op}} & & \sqsubseteq_{X_1}; \mathbf{Rel}(F)(R) & \xrightarrow{\varphi_1} & \sqsubseteq_{X_1} \\ & \searrow Fs^{-1} & \downarrow \varphi_2 & \lrcorner & \downarrow i_2^1 \\ & & FR & \xrightarrow{Fp_1} & FX_1 \end{array}$$

63 Obviously, s'' and s''^{-1} are each other's inverse, thus $\sqsubseteq_{X_1}; \mathbf{Rel}(F)(R)$ and
64 $\mathbf{Rel}(F)(R^{\text{op}}); \sqsubseteq_{X_1}^{\text{op}}$ are isomorphic.

$$\begin{aligned} Fp'_1 \cdot \varphi'_1 \cdot s'' \cdot \pi_1 &= Fp'_1 \cdot Fs \cdot \varphi_2 \cdot \pi_1 \\ &= Fp_2 \cdot \varphi_2 \cdot \pi_1 \\ &= i_1^2 \cdot \pi_2 \\ &= i_2'^2 \cdot s_2 \cdot \pi_2 \end{aligned}$$

65

$$\begin{array}{ccccc} \sqsubseteq_{X_1}; \mathbf{Rel}(F)(R); \sqsubseteq_{X_2} & \xrightarrow{\pi_1} & \sqsubseteq_{X_1}; \mathbf{Rel}(F)(R) & & \\ \pi_2 \downarrow & \searrow s' & \searrow s'' & & \\ & & \sqsubseteq_{X_2}^{\text{op}}; \mathbf{Rel}(F)(R^{\text{op}}); \sqsubseteq_{X_1}^{\text{op}} & \xrightarrow{\pi'_2} & \mathbf{Rel}(F)(R^{\text{op}}); \sqsubseteq_{X_1}^{\text{op}} \\ & \searrow s_2 & \downarrow \pi'_1 & \lrcorner & \downarrow \varphi'_1 \\ & & \sqsubseteq_{X_2}^{\text{op}} & \xrightarrow{i_2'^2} & FR^{\text{op}} \\ & & & & \downarrow Fp'_1 \\ & & & & FX_2 \end{array}$$

$$\begin{aligned} Fp_2 \cdot \varphi_2 \cdot s''^{-1} \cdot \pi'_2 &= Fp_2 \cdot Fs^{-1} \cdot \varphi'_1 \cdot \pi'_2 \\ &= Fp'_1 \cdot \varphi'_1 \cdot \pi'_2 \\ &= i_2'^2 \cdot \pi'_1 \\ &= i_1^2 \cdot s_2^{-1} \cdot \pi'_1 \end{aligned}$$

66

$$\begin{array}{ccccc}
\sqsubseteq_{X_2}^{\text{op}}; \mathbf{Rel}(F)(R^{\text{op}}); \sqsubseteq_{X_1}^{\text{op}} & \xrightarrow{\pi'_2} & \mathbf{Rel}(F)(R^{\text{op}}); \sqsubseteq_{X_1}^{\text{op}} & & \\
\downarrow \pi'_1 & \searrow s'^{-1} & \downarrow s''^{-1} & & \\
& \sqsubseteq_{X_1}; \mathbf{Rel}(F)(R); \sqsubseteq_{X_2} & \xrightarrow{\pi_1} & \sqsubseteq_{X_1}; \mathbf{Rel}(F)(R) & \\
& \downarrow \pi_2 & \lrcorner & \downarrow \varphi_2 & \\
& \sqsubseteq_{X_2} & \xrightarrow{i_1^2} & FX_2 & \\
& \xleftarrow{s_2^{-1}} & & & \\
& \sqsubseteq_{X_2}^{\text{op}} & & & \\
& \xrightarrow{\pi'_1} & & & \\
& \sqsubseteq_{X_2}^{\text{op}} & & & \\
& \xrightarrow{s_2^{-1}} & & & \\
& \sqsubseteq_{X_2} & & & \\
& \xrightarrow{i_1^2} & & & \\
& FX_2 & & &
\end{array}$$

So, we could prove that $\sqsubseteq_{X_1}; \mathbf{Rel}(F)(R); \sqsubseteq_{X_2}$ and $\sqsubseteq_{X_2}^{\text{op}}; \mathbf{Rel}(F)(R^{\text{op}}); \sqsubseteq_{X_1}^{\text{op}}$ are isomorphic. $(\sqsubseteq_{X_1}; \mathbf{Rel}(F)(R); \sqsubseteq_{X_2})^{\text{op}}$ is isomorphic to $\sqsubseteq_{X_1}; \mathbf{Rel}(F)(R); \sqsubseteq_{X_2}$ by definition, so it is also isomorphic with $\sqsubseteq_{X_2}^{\text{op}}; \mathbf{Rel}(F)(R^{\text{op}}); \sqsubseteq_{X_1}^{\text{op}}$. \square

67 **Proposition 1.6.** *Having $\sigma: R \rightarrow \sqsubseteq_{X_2}; \mathbf{Rel}(F)(R); \sqsubseteq_{X_1}$ and $\sigma^{\text{op}}: R^{\text{op}} \rightarrow \sqsubseteq_{X_1}$*
68 *; $\mathbf{Rel}(F)(R^{\text{op}}); \sqsubseteq_{X_2}$ gives rise to a morphism $\gamma: R \rightarrow \mathbf{Rel}(F)(R)$, and vice-versa.*

Proof.

$$\begin{array}{ccccccc}
R & \xrightarrow{p_1} & X_1 & & X_1 & & \\
\sigma \searrow & & \downarrow \alpha & & \downarrow \alpha & & \\
\sqsubseteq_{X_1}; \mathbf{Rel}(F)(R); \sqsubseteq_{X_2} & \xrightarrow{\pi_1} & \sqsubseteq_{X_1}; \mathbf{Rel}(F)(R) & \xrightarrow{\varphi_1} & \sqsubseteq_{X_1} & \xrightarrow{i_1^1} & FX_1 \\
& \lrcorner & \downarrow \varphi_2 & \lrcorner & \downarrow i_2^1 & \downarrow s_1 & \downarrow i_2'^1 \\
& & FR & \xrightarrow{Fp_1} & FX_1 & \xleftarrow{i_1'^1} & \sqsubseteq_{X_1}^{\text{op}} \\
& & \downarrow Fp_2 & \searrow Fp_1' & \downarrow Fp_2' & \downarrow \varphi_2' & \uparrow \varphi_2' \\
& & FX_2 & \xrightarrow{Fs} & FR^{\text{op}} & \xleftarrow{\varphi_1'} & \mathbf{Rel}(F)(R^{\text{op}}); \sqsubseteq_{X_1}^{\text{op}} \\
& & \downarrow i_2^2 & \downarrow i_2'^2 & \downarrow \pi_2' & \downarrow \pi_2' & \downarrow \pi_2' \\
& & \sqsubseteq_{X_2} & \xrightarrow{i_1^2} & FX_2 & \xleftarrow{\pi_1'} & \sqsubseteq_{X_2}^{\text{op}}; \mathbf{Rel}(F)(R^{\text{op}}); \sqsubseteq_{X_1}^{\text{op}} \\
& & \downarrow \beta & \downarrow \beta & \downarrow \beta & \downarrow \beta & \downarrow \beta \\
& & X_2 & \xrightarrow{\beta} & FX_2 & \xleftarrow{i_1'^2} & \sqsubseteq_{X_2}^{\text{op}} \\
& & & & \downarrow \beta & & \\
& & & & X_2 & & \\
& & & & \xleftarrow{p_1'} & & \\
& & & & R^{\text{op}} & & \\
& & & & \xleftarrow{\sigma^{\text{op}}} & & \\
& & & & \sqsubseteq_{X_1}^{\text{op}} & & \\
& & & & \xleftarrow{p_2'} & & \\
& & & & X_1 & &
\end{array}$$

69 (\Leftarrow) : We assume that we have the morphism $\gamma: R \rightarrow FR$ such that the following
70 diagram commutes:

$$\begin{array}{ccccc}
X_2 & \xleftarrow{p_2} & R & \xrightarrow{p_1} & X_1 \\
\beta \downarrow & & \downarrow \gamma & & \downarrow \alpha \\
FX_2 & \xleftarrow{Fp_2} & FR & \xrightarrow{Fp_1} & FX_1
\end{array} \tag{1}$$

71 Since \sqsubseteq_{X_1} and \sqsubseteq_{X_1} preorders, they each have a morphism refl that pre-composed
72 with their projections gives identity. As it is depicted in the following diagram
73 the pullback property of $\sqsubseteq_{X_1}; \mathbf{Rel}(F)(R)$ gives us $\sigma': R \rightarrow \sqsubseteq_{X_1}; \mathbf{Rel}(F)(R)$ in

74 the following commutative diagram:

$$\begin{array}{ccccc}
 R & \xrightarrow{p_1} & X_1 & \xrightarrow{\alpha} & FX_1 \\
 & \searrow \sigma' & & & \downarrow \text{refl} \\
 & & \sqsubseteq_{X_1}; \mathbf{Rel}(F)(R) & \xrightarrow{\varphi_1} & \sqsubseteq_{X_1} \\
 & \searrow \gamma & \downarrow \varphi_2 & \lrcorner & \downarrow i_2^1 \\
 & & FR & \xrightarrow{Fp_1} & FX_1
 \end{array} \quad (2) \quad \{\text{eq:diag-thm-sig'}\}$$

75 Then the pullback property of $\sqsubseteq_{X_1}; \mathbf{Rel}(F)(R); \sqsubseteq_{X_2}$ gives us the existence of
 76 $\sigma: R \rightarrow \sqsubseteq_{X_1}; \mathbf{Rel}(F)(R); \sqsubseteq_{X_2}$ in the following commutative diagram:

$$\begin{array}{ccccc}
 X_2 & \xleftarrow{p_2} & R & & \\
 \downarrow \beta & & \swarrow \sigma & \swarrow \sigma' & \\
 & & \sqsubseteq_{X_1}; \mathbf{Rel}(F)(R); \sqsubseteq_{X_2} & \xrightarrow{\pi_1} & \sqsubseteq_{X_1}; \mathbf{Rel}(F)(R) \\
 & & \downarrow \pi_2 & \lrcorner & \downarrow \varphi_2 \\
 & & & & FR \\
 & & & & \downarrow Fp_2 \\
 FX_2 & \xrightarrow{\text{refl}} & \sqsubseteq_{X_2} & \xrightarrow{i_1^2} & FX_2
 \end{array} \quad (3)$$

\{\text{eq:diag-thm-sig}\}

77 Now, we show that σ is a simulation:

$$\begin{aligned}
 i_1^1 \cdot \varphi_1 \cdot \pi_1 \cdot \sigma &= i_1^1 \cdot \varphi_1 \cdot \sigma' & // (3) \\
 &= i_1^1 \cdot \text{refl} \cdot \alpha \cdot p_1 & // (2) \\
 &= \alpha \cdot p_1
 \end{aligned}$$

78

$$\begin{aligned}
 i_2^2 \cdot \pi_2 \cdot \sigma &= i_2^2 \cdot \text{refl} \cdot \beta \cdot p_2 & // (3) \\
 &= \beta \cdot p_2
 \end{aligned}$$

79 Considering that $s: R \rightarrow R^{\text{op}}$ and $s': \sqsubseteq_{X_1}; \mathbf{Rel}(F)(R); \sqsubseteq_{X_2} \rightarrow \sqsubseteq_{X_2}^{\text{op}}$
 80 $; \mathbf{Rel}(F)(R^{\text{op}}); \sqsubseteq_{X_1}^{\text{op}}$ are swapping isomorphisms, We set $\sigma^{\text{op}}: R^{\text{op}} \rightarrow \sqsubseteq_{X_2}^{\text{op}}$
 81 $; \mathbf{Rel}(F)(R^{\text{op}}); \sqsubseteq_{X_1}^{\text{op}}$ to be $\sigma^{\text{op}} = s' \cdot \sigma \cdot s^{-1}$. Now, we show that σ' is a simu-
 82 lation:

$$\begin{aligned}
 i_2'^1 \cdot \varphi_2' \cdot \pi_2' \cdot \sigma^{\text{op}} &= i_2'^1 \cdot \varphi_2' \cdot \pi_2' \cdot s' \cdot \sigma \cdot s^{-1} \\
 &= i_1^1 \cdot \varphi_1 \cdot \pi_1 \cdot \sigma \cdot s^{-1} \\
 &= \alpha \cdot p_1 \cdot s^{-1}
 \end{aligned}$$

$$= \alpha \cdot p'_2$$

83

$$\begin{aligned} i_1'^2 \cdot \pi_1' \cdot \sigma^{\text{op}} &= \\ &= i_1'^2 \cdot \pi_1' \cdot s' \cdot \sigma \cdot s^{-1} \\ &= i_2^2 \cdot \pi_2 \cdot \sigma \cdot s^{-1} \\ &= \beta \cdot p_2 \cdot s^{-1} \\ &= \beta \cdot p_1' \end{aligned}$$

84 **1.2 Simulation with one relation composition**

85 We recall everything we had in the previous section. Although we want to work
86 with the functor that takes $R \subseteq X_1 \times X_2$ and gives $\mathbf{Rel}(F)(R); \sqsubseteq_{X_2}$.

$$\begin{array}{c} \mathbf{Rel}(F)(R); \sqsubseteq_{X_2} \xrightarrow{\pi_1} FR \xrightarrow{Fp_1} FX_1 \\ \pi_2 \downarrow \quad \lrcorner \quad \downarrow Fp_2 \\ \sqsubseteq_{X_2} \xrightarrow{i_1} FX_2 \\ i_2 \downarrow \\ FX_2 \end{array}$$

$$\begin{array}{ccccccc} & R & \xrightarrow{p_1} & X_1 & & & \\ & \searrow \sigma & & \downarrow \alpha & & & \\ & \mathbf{Rel}(F)(R); \sqsubseteq_{X_2} & \xrightarrow{\pi_1} & FR & \xrightarrow{Fp_1} & FX_1 & \xleftarrow{\alpha} X_1 \\ & \pi_2 \downarrow & \lrcorner & \downarrow Fp_2 & & \uparrow Fp_1 & \\ & \sqsubseteq_{X_2} & \xrightarrow{i_1} & FX_2 & \xleftarrow{Fp_2} & FR & \\ & i_2 \downarrow & & \uparrow i_1' & & \uparrow \pi_1' & \\ X_2 & \xrightarrow{\beta} & FX_2 & \xleftarrow{i_2'} & \sqsubseteq_{X_2}^{\text{op}} & \xleftarrow{\pi_2'} & \mathbf{Rel}(F)(R); \sqsubseteq_{X_2}^{\text{op}} \\ & \uparrow \beta & & & & & \swarrow \sigma^{\text{op}} \\ & X_2 & \xleftarrow{p_2} & R & & & \end{array}$$

88

89 **Proposition 1.7.** Assuming $R \subseteq X \times X$, then if we have $\sigma: R \rightarrow$
90 $\mathbf{Rel}(F)(R); \sqsubseteq_X$ as a simulation for R , and R is reflexive, then we have $\gamma: R \rightarrow$
91 $\mathbf{Rel}(F)(R)$ as a bisimulation for R , and vice-versa.

92 *Proof.* (\Rightarrow) :

$$\begin{aligned} Fp_2 \cdot \pi_1 \cdot \sigma &= \\ &= i_1 \cdot \pi_2 \cdot \sigma \\ &= i_2' \cdot s \cdot \pi_2 \cdot \sigma \\ &= \end{aligned}$$

1.3 Using Lax Pullbacks (Comma Objects) to Model Simulation

A big concern with this approach is that Comma Objects are defined in a 2-category, so we can not define them in **Set**, while our main inspirational example is coming from **Set**.

1.4 Working in Set First, Like Hughes and Jacobs

1.5 Choosing a suitable order for our setting

Maybe we can first choose a suitable order on $T(\Sigma_\vee \mu \Sigma \times D(\mu \Sigma, \mu \Sigma))$ and then prove that if a relation and its inverse is a simulation then it is a bisimulation as well. Maybe T being ω -continuous can give the ordering. It can be something easier that relates to termination as well! That if a term has a big-step evaluation, then it is bigger than or equal to any other term, and if it does not, then it is less than or equal to any other term.

2 Symmetric Simulation is Bisimulation

Definition 2.1 (Graph). In a category **C** a graph is a tuple (R, X) of the following form:

$$\begin{array}{ccc} & R & \\ p_1 \swarrow & & \searrow p_2 \\ X & & X \end{array}$$

Graphs over **C** form a category that we show by **Gra(C)**.

Definition 2.2 (Symmetric Graph). A graph (R, X) is symmetric iff there exists an endomorphism $s: R \rightarrow R$, such that the following diagram commutes

$$\begin{array}{ccccc} X & \xleftarrow{p_2} & R & \xrightarrow{p_1} & X \\ \text{id} \downarrow & & \downarrow s & & \downarrow \text{id} \\ X & \xleftarrow{p_1} & R & \xrightarrow{p_2} & X \end{array}$$

and $s \cdot s = \text{id}$. We call s a *swap* for R .

{lem:gra-sym}

Lemma 2.3. *Symmetry of a graphs over preserved a functor.*

Definition 2.4 (Relation). A relation in a category **C** is a graph (R, X) where $\langle p_1, p_2 \rangle: R \rightarrow X \times X$ is monic. Relations over **C** form a category that we show by **Rel(C)**.

Definition 2.5 (Jointly Monic). A pair of morphisms $p_1, p_2: R \rightarrow X$ is jointly monic iff for every pair of morphisms $f, g: A \rightarrow R$ assuming that $p_1 \cdot f = p_1 \cdot g$ and $p_2 \cdot f = p_2 \cdot g$ then $f = g$.

{prop:rel-joi-mon}

119 **Proposition 2.6.** *A graph (R, X) is a relation iff p_1 and p_2 are jointly monic.*

120 *Proof.* (\Rightarrow) : We assume that for morphisms $f, g: A \rightarrow R$ we have $p_1 \cdot f = p_1 \cdot g$ and
 121 $p_2 \cdot f = p_2 \cdot g$, and we want to prove that $f = g$. Assuming that $\pi_1, \pi_2: X \times X \rightarrow X$
 122 are projections of $X \times X$, then we have:

$$\begin{aligned} \langle p_1, p_2 \rangle \cdot f &= \langle p_1 \cdot f, p_2 \cdot f \rangle \\ &= \langle p_1 \cdot g, p_2 \cdot g \rangle \\ &= \langle p_1, p_2 \rangle \cdot g \end{aligned}$$

123 Since $\langle p_1, p_2 \rangle$ is monic, from $\langle p_1, p_2 \rangle \cdot f = \langle p_1, p_2 \rangle \cdot g$ we get $f = g$.

(\Leftarrow) : Assuming for some morphisms $f, g: A \rightarrow R$ we have $\langle p_1, p_2 \rangle \cdot f = \langle p_1, p_2 \rangle \cdot g$ we need to prove $f = g$. From $\langle p_1, p_2 \rangle \cdot f = \langle p_1, p_2 \rangle \cdot g$ we get $\langle p_1 \cdot f, p_2 \cdot f \rangle = \langle p_1 \cdot g, p_2 \cdot g \rangle$. Assuming that $\pi_1, \pi_2: X \times X \rightarrow X$ are projections of $X \times X$, then we have $\pi_1 \cdot \langle p_1 \cdot f, p_2 \cdot f \rangle = \pi_1 \cdot \langle p_1 \cdot g, p_2 \cdot g \rangle$, and then $p_1 \cdot f = p_1 \cdot g$. Similarly we also get $p_2 \cdot f = p_2 \cdot g$. So, since p_1 and p_2 are jointly monic, then we have $f = g$. \square

124 We need to work with endofunctors over \mathbf{C} that are lifted over $\mathbf{Rel}(\mathbf{C})$, for
 125 which we need to first define endofunctors lifted over $\mathbf{Gra}(\mathbf{C})$. Lifting from \mathbf{C}
 126 to $\mathbf{Gra}(\mathbf{C})$ is easy. For $F: \mathbf{C} \rightarrow \mathbf{C}$ we define $F_{\mathbf{Gra}}: \mathbf{Gra}(\mathbf{C}) \rightarrow \mathbf{Gra}(\mathbf{C})$ as a
 127 functor that takes a graph (R, X) , and gives (FR, FX) , where F is also applied
 128 on legs of the graph, i.e., $p_1, p_2: R \rightarrow X$, so, we get the following graph:

$$\begin{array}{ccc} & FR & \\ Fp_1 \swarrow & & \searrow Fp_2 \\ FX & & FX \end{array}$$

129 This lifting does not work for \mathbf{Rel} . As an example, if we set F to be the powerset
 130 functor \mathcal{P} , then $(\mathcal{P}R, \mathcal{P}X)$ is not necessarily a relation anymore. For example, if
 131 we take $R = \{(1, 0), (0, 1), (0, 0), (1, 1)\}$, then taking $\{\{(1, 0), (0, 1), (0, 0), (1, 1)\}\}$
 132 and $\{\{(1, 0), (0, 1), (0, 0)\}\}$ as elements of $\mathcal{P}R$, the morphism $\langle \mathcal{P}p_1, \mathcal{P}p_2 \rangle$ maps
 133 them both to $(\{0, 1\}, \{0, 1\})$ so it is not monic.

134 To cope with this, we assume the following epi-mono decomposition for
 135 $(R, X) \in \mathbf{Rel}(\mathbf{C})$:

$$\begin{array}{ccccc} & & \langle p_1, p_2 \rangle & & \\ & \searrow & \curvearrowright & \searrow & \\ R & \xrightarrow{e_R} & R^\dagger & \xrightarrow{\langle p_1^\dagger, p_2^\dagger \rangle} & X \times X \end{array}$$

136 We can define $(-)^{\dagger}$ as a functor from $\mathbf{Gra}(\mathbf{C}) \rightarrow \mathbf{Rel}(\mathbf{C})$, then we define
 137 $F_{\mathbf{Rel}}: \mathbf{Rel}(\mathbf{C}) \rightarrow \mathbf{Rel}(\mathbf{C})$ to take (R, X) to the following relation:

$$\begin{array}{ccc} & (FR)^{\dagger} & \\ (Fp_1)^{\dagger} \swarrow & \downarrow & \searrow (Fp_2)^{\dagger} \\ FX & \langle (Fp_1)^{\dagger}, (Fp_2)^{\dagger} \rangle & FX \\ & \downarrow & \\ & FX \times FX & \end{array}$$

138
 {lem:norm-simp}

139 **Lemma 2.7.** *Assuming that we have the following commutative diagram:*

$$\begin{array}{ccccc} X & \xleftarrow{p_1} & R & \xrightarrow{p_2} & X \\ \alpha \downarrow & & \downarrow \sigma & & \downarrow \alpha \\ FX & \xleftarrow{Fp_1} & FR & \xrightarrow{Fp_2} & FX \end{array}$$

140 *Then there exists $\sigma^{\dagger}: R \rightarrow (FR)^{\dagger}$ in the following diagram that is also commu-*
 141 *tative:*

$$\begin{array}{ccccc} X & \xleftarrow{p_1} & R & \xrightarrow{p_2} & X \\ \alpha \downarrow & & \downarrow \sigma^{\dagger} & & \downarrow \alpha \\ FX & \xleftarrow{Fp_1^{\dagger}} & (FR)^{\dagger} & \xrightarrow{Fp_2^{\dagger}} & FX \end{array}$$

142 *Proof.* The proof is trivial considering that $\sigma^{\dagger} = e_{FR} \cdot \sigma$, where e_{FR} is the
 143 epimorphism in the epi-mono factorization of $\langle Fp_1, Fp_2 \rangle$, as depicted in the
 144 following diagram:

$$\begin{array}{ccccc} X & \xleftarrow{p_1} & R & \xrightarrow{p_2} & X \\ \alpha \downarrow & & \downarrow \sigma & & \downarrow \alpha \\ FX & \xleftarrow{Fp_1} & FR & \xrightarrow{Fp_2} & FX \\ \text{id} \downarrow & & \downarrow e_{FR} & & \downarrow \text{id} \\ FX & \xleftarrow{Fp_1^{\dagger}} & (FR)^{\dagger} & \xrightarrow{Fp_2^{\dagger}} & FX \end{array}$$

□

145 We show this relation with $F_{\mathbf{Rel}}(R, X)$.

{def:sim} **Definition 2.8 (Simulation).** A coalgebra $\sigma: R \rightarrow (FR)^{\dagger}$ is a simulation over
 147 the F -coalgebra $\alpha: X \rightarrow FX$ iff the following diagram is lax-commutative:

$$\begin{array}{ccccc} X & \xleftarrow{p_1} & R & \xrightarrow{p_2} & X \\ \alpha \downarrow & \sqsubseteq & \downarrow \sigma & \sqsubseteq & \downarrow \alpha \\ FX & \xleftarrow{(Fp_1)^{\dagger}} & (FR)^{\dagger} & \xrightarrow{(Fp_2)^{\dagger}} & FX \end{array} \quad (4)$$

{eq:diag-lax-sim}

{def:bisim}
149

Definition 2.9 (Bisimulation). The morphism σ in [Definition 2.8](#) is a bisimulation iff the mentioned diagram is fully commutative.

150 **Remark 2.10.** The mentioned definition of bisimulation is actually, the classical
151 one in the literature that is to have the following commutative diagram:

$$\begin{array}{ccccc} X & \xleftarrow{p_1} & R & \xrightarrow{p_2} & X \\ \alpha \downarrow & & \downarrow \sigma & & \downarrow \alpha \\ FX & \xleftarrow{(Fp_1)^\dagger} & (FR)^\dagger & \xrightarrow{(Fp_2)^\dagger} & FX \end{array}$$

152 It may look different because we have FX and not $(FX)^\dagger$, but they are the
153 same. An object $X \in \mathbf{Obj}(\mathbf{C})$ is $(X, X) \in \mathbf{Rel}(\mathbf{C})$ having id as its legs. Meaning
154 that the $(FX)^\dagger = FX$.

155 **Proposition 2.11.** Assuming that we have a bisimulation σ for R , we have the
156 following equation:

{prop:iff-sim-bsim}

$$\sigma \cdot s = (Fs)^\dagger \cdot \sigma$$

157 *Proof.* We recall that by [Lemma 2.3](#), $F_{\mathbf{Rel}}(R, X)$ is symmetric with the swap
158 $(Fs)^\dagger$. Assuming that σ is a bisimulation, we have the following commutative
159 diagram:

$$\begin{array}{ccccc} & & R & & \\ & \swarrow p_2 & \downarrow s & \searrow p_1 & \\ X & \xleftarrow{p_1} & R & \xrightarrow{p_2} & X \\ \alpha \downarrow & & \downarrow \sigma & & \downarrow \alpha \\ FX & \xleftarrow{(Fp_1)^\dagger} & (FR)^\dagger & \xrightarrow{(Fp_2)^\dagger} & FX \\ & \nwarrow (Fp_2)^\dagger & \downarrow (Fs)^\dagger & \nearrow (Fp_1)^\dagger & \end{array}$$

(5) {eq:diag-sym-rel}

160 And it entails that the following diagrams are also commutative:

$$\begin{array}{ccccc} X & \xleftarrow{p_2} & R & \xrightarrow{p_1} & X \\ \alpha \downarrow & & \downarrow \sigma \cdot s & & \downarrow \alpha \\ FX & \xleftarrow{(Fp_1)^\dagger} & (FR)^\dagger & \xrightarrow{(Fp_2)^\dagger} & FX \end{array} \quad \begin{array}{ccccc} X & \xleftarrow{p_2} & R & \xrightarrow{p_1} & X \\ \alpha \downarrow & & \downarrow (Fs)^\dagger \cdot \sigma & & \downarrow \alpha \\ FX & \xleftarrow{(Fp_1)^\dagger} & (FR)^\dagger & \xrightarrow{(Fp_2)^\dagger} & FX \end{array}$$

So, since $(Fp_1)^\dagger$ and $(Fp_2)^\dagger$ are jointly monic (because $F_{\mathbf{Rel}}(R, X)$ is a relation and [Proposition 2.6](#)) we have $\sigma \cdot s = (Fs)^\dagger \cdot \sigma$. \square

161 **Corollary 2.12.** Assuming σ_1 and σ_2 are simulations of type $R \rightarrow (FR)^\dagger$, and
162 R is symmetric and both σ_1 and σ_2 satisfy the following property:

$$(Fs)^\dagger \cdot \sigma = \sigma \cdot s$$

163 Then $\sigma_1 = \sigma_2$.

164 *Proof.* As the mentioned property is equivalent with σ being a bisimulation, and
 165 bisimulation is unique, then $\sigma_1 = \sigma_2$.

166 Now, we give a counter example of a symmetric relation on **Set** that is a
 167 simulation according to [Definition 2.8](#), i.e, exists the morphism σ that com-
 168 mutes laxly in (4), but σ is not a coalgebraic bisimulation, although the
 169 relation that we give is clearly a bisimulation in the classic sense. We set
 170 $R = \{(A, B), (B, A), (C_1, C_2), (C_2, C_1), (C'_2, C_2), (C_2, C'_2), (C_2, C_2)\}$, $F = \mathbf{Id}$,
 171 $\sqsubseteq = \Delta \cup \{(C_1, C_2), (C_2, C'_2)\}$, and the coalgebra α is defined with the follow-
 172 ing set of reductions:

$$A \rightarrow C_1 \quad B \rightarrow C_2 \quad C_1 \rightarrow C_1 \quad C_2 \rightarrow C_2 \quad C'_2 \rightarrow C_2$$

173 And finally, we define σ as follows:

$$\sigma(w) = \begin{cases} (\alpha \cdot p_1(w), \alpha \cdot p_2(w)) & w \neq (B, A) \\ (C'_2, C_2) & w = (B, A) \end{cases}$$

174 It is easy to check that the conditions $\alpha \cdot p_1 \sqsubseteq (Fp_1)^\dagger \cdot \sigma$ and $(Fp_2)^\dagger \cdot \sigma \sqsubseteq \alpha \cdot p_2$
 175 are satisfied. For every $w \in R$ if $w \neq (B, A)$ then for $i \in \{1, 2\}$, we have $\alpha \cdot p_i =$
 176 $(Fp_i)^\dagger \cdot \sigma$, and for $w = (B, A)$ we have $\alpha \cdot p_1(B, A) = C_2 \sqsubseteq C'_2 = (Fp_1)^\dagger \cdot \sigma(B, A)$,
 177 and $\alpha \cdot p_2(B, A) = C_1 \sqsubseteq C_2 = (Fp_2)^\dagger \cdot \sigma(B, A)$. And σ is not a coalgebraic
 178 bisimulation as $\alpha \cdot p_1(B, A) = C_2 \neq C'_2 = (Fp_1)^\dagger \cdot \sigma(B, A)$.

179 An interesting question would be to find out what conditions σ should have
 180 (maybe we have the answer to this! [Proposition 2.11](#)), or how it should be con-
 181 structed (perhaps based on a given poset) so that it will also be a coalgebraic
 182 bisimulation if R is symmetric. Another avenue would be to give another defini-
 183 tion for simulation that does not have this issue.

184 Well! This counter example does not work! Because the described order \sqsubseteq
 185 does not satisfy the condition mentioned in Jacobs's paper. The condition is that
 186 the order on FX should satisfy the property that for a morphism $f: X \rightarrow Y$ the
 187 morphism $Ff: FX \rightarrow FY$ preserves \sqsubseteq . Probably, the only poset that has this
 188 property for \mathbf{Id} is Δ . If there is a counter-example, it is true for another functor.

189 (But still!) We have a counter-example for a symmetric relation R that has a
 190 witness to be a simulation, but that morphism does not serve as a witness for
 191 R to be a bisimulation. In the category of sets we assume that $F = \mathcal{P}$, and take
 192 $R = \{(1, 2), (2, 1), (1, 3), (3, 1)\}$, and $X = \{1, 2, 3\}$. $\alpha(x) = X$ for every $x \in X$,
 193 and σ is defined as below:

$$\sigma(w) = \begin{cases} (X, X) & w \neq (1, 3) \\ (X, X \setminus \{2\}) & w = (1, 3) \end{cases}$$

194 In this scenario, σ is a witness for R to be a simulation, but it is not a witness
 195 for R to be a bisimulation. σ is a witness for R to be a simulation since for
 196 every $w \in R$ we have $\alpha(p_1(w)) \sqsubseteq ((\mathcal{P}p_1)^\dagger(\sigma(w))) = X$. Also, for every $w \in R$,
 197 $((\mathcal{P}p_2)^\dagger(\sigma(w))) \sqsubseteq \alpha(p_2(w)) = X$. But it is not a bisimulation, since $\alpha(p_2(1, 3)) =$
 198 $X \setminus \{2\} \neq X = \alpha(p_1(1, 3))$.

199 **Example 2.13.** And another counter-example!!! Assume that $F = \mathcal{P}$, and take
 200 $R = X \times X \setminus \{(1, 3), (3, 1)\}$, and $X = \{1, 2, 3\}$. α is defined as below:

$$\alpha(x) = \begin{cases} \{1, 2\} & x = 1 \\ \{2, 3\} & x = 2 \\ \{3\} & x = 3 \end{cases}$$

201 And σ_1 is defined as below:

$$\sigma_1(w) = \begin{cases} \{(1, 2), (2, 2)\} & w = (1, 2) \\ \{(2, 1), (3, 2)\} & w = (2, 1) \\ \{(1, 2), (2, 1)\} & w = (1, 1) \\ \{(2, 2), (3, 3)\} & w \in \{(2, 2), (3, 2)\} \\ \{(2, 3), (3, 3)\} & w = (2, 3) \\ \{(3, 3)\} & w = (3, 3) \end{cases}$$

202

$$(\mathcal{P}p_1)^\dagger \cdot \sigma_1(w) = \begin{cases} \{1, 2\} & w = (1, 2) \\ \{2, 3\} & w = (2, 1) \\ \{1, 2\} & w = (1, 1) \\ \{2, 3\} & w \in \{(2, 2), (3, 2)\} \\ \{2, 3\} & w = (2, 3) \\ \{3\} & w = (3, 3) \end{cases} \quad (\mathcal{P}p_2)^\dagger \cdot \sigma_1(w) = \begin{cases} \{2\} & w = (1, 2) \\ \{1, 2\} & w = (2, 1) \\ \{1, 2\} & w = (1, 1) \\ \{2, 3\} & w \in \{(2, 2), (3, 2)\} \\ \{3\} & w = (2, 3) \\ \{3\} & w = (3, 3) \end{cases} \quad \blacksquare$$

203

$$\sigma'_1(w) = \begin{cases} \{(1, 2), (2, 2), (2, 3)\} & w = (1, 2) \\ \{(2, 1), (2, 2), (3, 2)\} & w = (2, 1) \\ \{(1, 2), (2, 1)\} & w = (1, 1) \\ \{(2, 2), (3, 3)\} & w = (2, 2) \\ \{(2, 2), (3, 3), (3, 2)\} & w = (3, 2) \\ \{(2, 3), (2, 2), (3, 3)\} & w = (2, 3) \\ \{(3, 3)\} & w = (3, 3) \end{cases}$$

204

$$(\mathcal{P}p_1)^\dagger \cdot \sigma'_1(w) = \begin{cases} \{1, 2\} & w = (1, 2) \\ \{2, 3\} & w = (2, 1) \\ \{1, 2\} & w = (1, 1) \\ \{2, 3\} & w = (2, 2) \\ \{2, 3\} & w = (3, 2) \\ \{2, 3\} & w = (2, 3) \\ \{3\} & w = (3, 3) \end{cases} \quad (\mathcal{P}p_2)^\dagger \cdot \sigma'_1(w) = \begin{cases} \{2, 3\} & w = (1, 2) \\ \{1, 2\} & w = (2, 1) \\ \{1, 2\} & w = (1, 1) \\ \{2, 3\} & w = (2, 2) \\ \{2, 3\} & w = (3, 2) \\ \{2, 3\} & w = (2, 3) \\ \{3\} & w = (3, 3) \end{cases}$$

205 σ'_1 is not a simulation!

$$\sigma''_1(w) = \begin{cases} \{(1, 2)\} & w = (1, 2) \\ \{(2, 1)\} & w = (2, 1) \\ \{(1, 2), (2, 1)\} & w = (1, 1) \\ \{(2, 2), (3, 3)\} & w = (2, 2) \\ \{(3, 3)\} & w = (3, 2) \\ \{(3, 3)\} & w = (2, 3) \\ \{(3, 3)\} & w = (3, 3) \end{cases}$$

$$\begin{aligned} \sigma_1 &\sqsubseteq \sigma'_1 \\ \sigma_3 &\sqsubseteq \sigma'_1 \\ \beta &= \sigma'_1 \end{aligned}$$

206

$$(\mathcal{P}s)^\dagger \cdot \sigma_1 \cdot s(w) = \begin{cases} \{(1, 2), (2, 3)\} & w = (1, 2) \\ \{(2, 1), (2, 2)\} & w = (2, 1) \\ \{(1, 2), (2, 1)\} & w = (1, 1) \\ \{(2, 2), (3, 3)\} & w = (2, 2) \\ \{(3, 2), (3, 3)\} & w = (3, 2) \\ \{(2, 2), (3, 3)\} & w = (2, 3) \\ \{(3, 3)\} & w = (3, 3) \end{cases}$$

207

$$(\mathcal{P}p_1)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma_1 \cdot s(w) = \begin{cases} \{1, 2\} & w = (1, 2) \\ \{2\} & w = (2, 1) \\ \{1, 2\} & w = (1, 1) \\ \{2, 3\} & w = (2, 2) \\ \{3\} & w = (3, 2) \\ \{2, 3\} & w = (2, 3) \\ \{3\} & w = (3, 3) \end{cases} \quad (\mathcal{P}p_2)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma_1 \cdot s(w) = \begin{cases} \{2, 3\} & w = (1, 2) \\ \{1, 2\} & w = (2, 1) \\ \{1, 2\} & w = (1, 1) \\ \{2, 3\} & w = (2, 2) \\ \{2, 3\} & w = (3, 2) \\ \{2, 3\} & w = (2, 3) \\ \{3\} & w = (3, 3) \end{cases} \blacksquare$$

208 In this scenario, σ_1 is a simulation, but it is not a bisimulation. σ'_1 , σ''_1 and
 209 $(\mathcal{P}s)^\dagger \cdot \sigma_1 \cdot s$ are neither. We can not make σ_1 bigger here to make it a bisimulation
 210 as $\alpha \cdot p_1(3, 2) = \{3\} \subsetneq \{2, 3\} = (\mathcal{P}p_1)^\dagger \cdot \sigma_1(3, 2)$.

211 The following is also a simulation and not a bisimulation:

$$\sigma_2(w) = \begin{cases} \{(1, 2), (2, 2)\} & w = (1, 2) \\ \{(2, 1), (3, 2)\} & w = (2, 1) \\ \{(1, 2), (2, 1)\} & w = (1, 1) \\ \{(2, 2), (3, 3)\} & w = (2, 2) \\ \{(2, 3), (3, 3)\} & w = (2, 3) \\ \{(3, 3)\} & w \in \{(3, 2), (3, 3)\} \end{cases}$$

212

$$(\mathcal{P}p_1)^\dagger \cdot \sigma_2(w) = \begin{cases} \{1, 2\} & w = (1, 2) \\ \{2, 3\} & w = (2, 1) \\ \{1, 2\} & w = (1, 1) \\ \{2, 3\} & w = (2, 2) \\ \{2, 3\} & w = (2, 3) \\ \{3\} & w \in \{(3, 2), (3, 3)\} \end{cases} \quad (\mathcal{P}p_2)^\dagger \cdot \sigma_2(w) = \begin{cases} \{2\} & w = (1, 2) \\ \{1, 2\} & w = (2, 1) \\ \{1, 2\} & w = (1, 1) \\ \{2, 3\} & w = (2, 2) \\ \{3\} & w = (2, 3) \\ \{3\} & w \in \{(3, 2), (3, 3)\} \end{cases} \blacksquare$$

213

$$\sigma'_2(w) = \begin{cases} \{(1, 2), (2, 2), (2, 3)\} & w = (1, 2) \\ \{(2, 1), (2, 2), (3, 2)\} & w = (2, 1) \\ \{(1, 2), (2, 1)\} & w = (1, 1) \\ \{(2, 2), (3, 3)\} & w = (2, 2) \\ \{(3, 3), (3, 2)\} & w = (3, 2) \\ \{(2, 3), (3, 3)\} & w = (2, 3) \\ \{(3, 3)\} & w = (3, 3) \end{cases}$$

214

$$(\mathcal{P}p_1)^\dagger \cdot \sigma'_2(w) = \begin{cases} \{1, 2\} & w = (1, 2) \\ \{2, 3\} & w = (2, 1) \\ \{1, 2\} & w = (1, 1) \\ \{2, 3\} & w = (2, 2) \\ \{3\} & w = (3, 2) \\ \{2, 3\} & w = (2, 3) \\ \{3\} & w = (3, 3) \end{cases} \quad (\mathcal{P}p_2)^\dagger \cdot \sigma'_2(w) = \begin{cases} \{2, 3\} & w = (1, 2) \\ \{1, 2\} & w = (2, 1) \\ \{1, 2\} & w = (1, 1) \\ \{2, 3\} & w = (2, 2) \\ \{2, 3\} & w = (3, 2) \\ \{3\} & w = (2, 3) \\ \{3\} & w = (3, 3) \end{cases}$$

215

$$(\mathcal{P}s)^\dagger \cdot \sigma_2 \cdot s(w) = \begin{cases} \{(1, 2), (2, 3)\} & w = (1, 2) \\ \{(2, 1), (2, 2)\} & w = (2, 1) \\ \{(1, 2), (2, 1)\} & w = (1, 1) \\ \{(2, 2), (3, 3)\} & w = (2, 2) \\ \{(3, 3), (3, 2)\} & w = (3, 2) \\ \{(3, 3)\} & w = (2, 3) \\ \{(3, 3)\} & w = (3, 3) \end{cases}$$

216

$$(\mathcal{P}p_1)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma_2 \cdot s(w) = \begin{cases} \{1, 2\} & w = (1, 2) \\ \{2\} & w = (2, 1) \\ \{1, 2\} & w = (1, 1) \\ \{2, 3\} & w = (2, 2) \\ \{3\} & w = (3, 2) \\ \{2\} & w = (2, 3) \\ \{3\} & w = (3, 3) \end{cases} \quad (\mathcal{P}p_2)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma_2 \cdot s(w) = \begin{cases} \{2, 3\} & w = (1, 2) \\ \{1, 2\} & w = (2, 1) \\ \{1, 2\} & w = (1, 1) \\ \{2, 3\} & w = (2, 2) \\ \{2, 3\} & w = (3, 2) \\ \{3\} & w = (2, 3) \\ \{3\} & w = (3, 3) \end{cases}$$

217 σ_2 is a simulation, σ'_2 is a bisimulation, and $(\mathcal{P}s)^\dagger \cdot \sigma_2 \cdot s$ is neither. The following
 218 is both a simulation and a bisimulation:

$$\sigma_3(w) = \begin{cases} \{(1, 2), (2, 2), (2, 3)\} & w = (1, 2) \\ \{(2, 1), (2, 2), (3, 2)\} & w = (2, 1) \\ \{(1, 2), (2, 1)\} & w = (1, 1) \\ \{(2, 2), (3, 3)\} & w = (2, 2) \\ \{(2, 3), (3, 3)\} & w = (2, 3) \\ \{(3, 2), (3, 3)\} & w = (3, 2) \\ \{(3, 3)\} & w = (3, 3) \end{cases}$$

219

$$(\mathcal{P}p_1)^\dagger \cdot \sigma_3(w) = \begin{cases} \{1, 2\} & w = (1, 2) \\ \{2, 3\} & w = (2, 1) \\ \{1, 2\} & w = (1, 1) \\ \{2, 3\} & w = (2, 2) \\ \{2, 3\} & w = (2, 3) \\ \{3\} & w = (3, 2) \\ \{3\} & w = (3, 3) \end{cases} \quad (\mathcal{P}p_2)^\dagger \cdot \sigma_3(w) = \begin{cases} \{2, 3\} & w = (1, 2) \\ \{1, 2\} & w = (2, 1) \\ \{1, 2\} & w = (1, 1) \\ \{2, 3\} & w = (2, 2) \\ \{3\} & w = (2, 3) \\ \{2, 3\} & w = (3, 2) \\ \{3\} & w = (3, 3) \end{cases}$$

220 The following is also a simulation and not a bisimulation:

$$\sigma_4(w) = \begin{cases} \{(1, 2), (2, 2), (3, 2), (2, 3), (3, 3)\} & w = (1, 2) \\ \{(1, 1), (2, 1), (1, 2), (2, 2), (3, 2)\} & w = (2, 1) \\ \{(1, 2), (2, 1)\} & w = (1, 1) \\ \{(2, 2), (3, 3)\} & w = (2, 2) \\ \{(2, 3), (3, 3)\} & w = (2, 3) \\ \{(1, 2), (2, 2), (3, 2), (2, 3), (3, 3)\} & w = (3, 2) \\ \{(3, 3)\} & w = (3, 3) \end{cases}$$

221

$$(\mathcal{P}p_1)^\dagger \cdot \sigma_4(w) = \begin{cases} \{1, 2, 3\} & w = (1, 2) \\ \{1, 2, 3\} & w = (2, 1) \\ \{1, 2\} & w = (1, 1) \\ \{2, 3\} & w = (2, 2) \\ \{2, 3\} & w = (2, 3) \\ \{1, 2, 3\} & w = (3, 2) \\ \{3\} & w = (3, 3) \end{cases} \quad (\mathcal{P}p_2)^\dagger \cdot \sigma_4(w) = \begin{cases} \{2, 3\} & w = (1, 2) \\ \{1, 2\} & w = (2, 1) \\ \{1, 2\} & w = (1, 1) \\ \{2, 3\} & w = (2, 2) \\ \{3\} & w = (2, 3) \\ \{2, 3\} & w = (3, 2) \\ \{3\} & w = (3, 3) \end{cases}$$

222

$$\sigma_4''(w) = \begin{cases} \{(1, 2), (2, 2), (2, 3)\} & w = (1, 2) \\ \{(2, 1), (2, 2), (3, 2)\} & w = (2, 1) \\ \{(1, 2), (2, 1)\} & w = (1, 1) \\ \{(2, 2), (3, 3)\} & w = (2, 2) \\ \{(2, 3), (3, 3)\} & w = (2, 3) \\ \{(3, 2), (3, 3)\} & w = (3, 2) \\ \{(3, 3)\} & w = (3, 3) \end{cases}$$

223

$$(\mathcal{P}p_1)^\dagger \cdot \sigma_4''(w) = \begin{cases} \{1, 2\} & w = (1, 2) \\ \{2, 3\} & w = (2, 1) \\ \{1, 2\} & w = (1, 1) \\ \{2, 3\} & w = (2, 2) \\ \{2, 3\} & w = (2, 3) \\ \{3\} & w = (3, 2) \\ \{3\} & w = (3, 3) \end{cases} \quad (\mathcal{P}p_2)^\dagger \cdot \sigma_4''(w) = \begin{cases} \{2, 3\} & w = (1, 2) \\ \{1, 2\} & w = (2, 1) \\ \{1, 2\} & w = (1, 1) \\ \{2, 3\} & w = (2, 2) \\ \{3\} & w = (2, 3) \\ \{2, 3\} & w = (3, 2) \\ \{3\} & w = (3, 3) \end{cases}$$

224

$$(\mathcal{P}s)^\dagger \cdot \sigma_4 \cdot s(w) = \begin{cases} \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3)\} & w = (1, 2) \\ \{(2, 1), (2, 2), (2, 3), (3, 2), (3, 3)\} & w = (2, 1) \\ \{(1, 2), (2, 1)\} & w = (1, 1) \\ \{(2, 2), (3, 3)\} & w = (2, 2) \\ \{(2, 1), (2, 2), (2, 3), (3, 2), (3, 3)\} & w = (2, 3) \\ \{(3, 2), (3, 3)\} & w = (3, 2) \\ \{(3, 3)\} & w = (3, 3) \end{cases}$$

225

$$(\mathcal{P}p_1)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma_4 \cdot s(w) = \begin{cases} \{1, 2\} & w = (1, 2) \\ \{2, 3\} & w = (2, 1) \\ \{1, 2\} & w = (1, 1) \\ \{2, 3\} & w = (2, 2) \\ \{2, 3\} & w = (2, 3) \\ \{3\} & w = (3, 2) \\ \{3\} & w = (3, 3) \end{cases} \quad (\mathcal{P}p_2)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma_4 \cdot s(w) = \begin{cases} \{1, 2, 3\} & w = (1, 2) \\ \{1, 2, 3\} & w = (2, 1) \\ \{1, 2\} & w = (1, 1) \\ \{2, 3\} & w = (2, 2) \\ \{1, 2, 3\} & w = (2, 3) \\ \{2, 3\} & w = (3, 2) \\ \{3\} & w = (3, 3) \end{cases} \blacksquare$$

226 σ_4 is a simulation, σ_4'' is a bisimulation, and $(\mathcal{P}s)^\dagger \cdot \sigma_4 \cdot s$ is neither.

227 The following is also a simulation and not a bisimulation:

$$\sigma_5(w) = \begin{cases} \{(1, 2), (2, 2)\} & w = (1, 2) \\ \{(2, 2), (3, 2)\} & w = (2, 1) \\ \{(1, 1), (2, 1)\} & w = (1, 1) \\ \{(2, 2), (3, 2)\} & w = (2, 2) \\ \{(2, 3), (3, 3)\} & w = (2, 3) \\ \{(3, 3)\} & w = (3, 2) \\ \{(3, 3)\} & w = (3, 3) \end{cases}$$

228

$$(\mathcal{P}p_1)^\dagger \cdot \sigma_5(w) = \begin{cases} \{1, 2\} & w = (1, 2) \\ \{2, 3\} & w = (2, 1) \\ \{1, 2\} & w = (1, 1) \\ \{2, 3\} & w = (2, 2) \\ \{2, 3\} & w = (2, 3) \\ \{3\} & w = (3, 2) \\ \{3\} & w = (3, 3) \end{cases} \quad (\mathcal{P}p_2)^\dagger \cdot \sigma_5(w) = \begin{cases} \{2\} & w = (1, 2) \\ \{2\} & w = (2, 1) \\ \{1\} & w = (1, 1) \\ \{2\} & w = (2, 2) \\ \{3\} & w = (2, 3) \\ \{3\} & w = (3, 2) \\ \{3\} & w = (3, 3) \end{cases}$$

229 The following is also a simulation and not a bisimulation:

$$\sigma_6(w) = \begin{cases} \{(1, 2), (2, 2)\} & w = (1, 2) \\ \{(2, 1), (3, 1)\} & w = (2, 1) \\ \{(1, 2), (2, 1)\} & w = (1, 1) \\ \{(2, 3), (3, 3)\} & w = (2, 2) \\ \{(2, 3), (3, 3)\} & w = (2, 3) \\ \{(3, 2)\} & w = (3, 2) \\ \{(3, 3)\} & w = (3, 3) \end{cases}$$

230

$$(\mathcal{P}p_1)^\dagger \cdot \sigma_6(w) = \begin{cases} \{1, 2\} & w = (1, 2) \\ \{2, 3\} & w = (2, 1) \\ \{1, 2\} & w = (1, 1) \\ \{2, 3\} & w = (2, 2) \\ \{2, 3\} & w = (2, 3) \\ \{3\} & w = (3, 2) \\ \{3\} & w = (3, 3) \end{cases} \quad (\mathcal{P}p_2)^\dagger \cdot \sigma_6(w) = \begin{cases} \{2\} & w = (1, 2) \\ \{1\} & w = (2, 1) \\ \{2\} & w = (1, 1) \\ \{3\} & w = (2, 2) \\ \{3\} & w = (2, 3) \\ \{2\} & w = (3, 2) \\ \{3\} & w = (3, 3) \end{cases}$$

231

$$\sigma'_2 = \sigma'_5 = \sigma'_6 = \sigma_3 = \sigma''_4$$

232 If we define \sqsubseteq on simulations as

$$\sigma_1 \sqsubseteq \sigma_2 \iff \forall x_1, x_2 \in X, (\mathcal{P}p_i)^\dagger \cdot \sigma_1(x_1, x_2) \subseteq (\mathcal{P}p_i)^\dagger \cdot \sigma_2(x_1, x_2)$$

233

234 **Lemma 2.14.** $(Hom(R, (\mathcal{P}R)^\dagger), \sqsubseteq)$ is a poset.

235 *Proof.* Reflexivity and transitivity are obvious. We need to prove anti-symmetry.

236 **TODO: Finish!**

237 Then we have

$$\begin{array}{ccccc}
 \sigma_6 & & \sigma_1 & & \\
 & \sqsubset & & \sqsubset & \\
 \sigma_5 & \sqsubseteq & \sigma_2 & \sqsubseteq & \sigma_3 & \sqsubseteq & \sigma_4
 \end{array}$$

238 We recall that in the above diagram σ_3 is a bisimulation, and the rest are simu-
 239 lations.

240 **Definition 2.15.** We define \sqcup and \sqcap on morphisms as follows:

$$\begin{aligned}
 & \forall x_1, x_2 \in X, \\
 & \sigma_1 \sqcup \sigma_2(x_1, x_2) = \sigma_1(x_1, x_2) \cup \sigma_2(x_1, x_2), \\
 & \sigma_1 \sqcap \sigma_2(x_1, x_2) = (\mathcal{P}p_1)^\dagger \cdot \sigma_1(x_1, x_2) \cap (\mathcal{P}p_1)^\dagger \cdot \sigma_2(x_1, x_2) \times (\mathcal{P}p_2)^\dagger \cdot \sigma_1(x_1, x_2) \cap (\mathcal{P}p_2)^\dagger \cdot \sigma_2(x_1, x_2). \blacksquare
 \end{aligned}$$

{def:join-meet}
{lem:proj-dist-set}

241 **Lemma 2.16.** For relations R_1 and R_2 the following equation holds:

$$(\mathcal{P}p_i)(R_1 \cup R_2) = (\mathcal{P}p_i)(R_1) \cup (\mathcal{P}p_i)(R_2)$$

242 *Proof.* We prove the lemma for the case that $i = 1$. The proof is the same for
 243 $i = 2$. Assuming $x_1 \in (\mathcal{P}p_1)^\dagger(R_1 \cup R_2)$ then exists x_2 that $(x_1, x_2) \in R_1 \cup R_2$,
 244 thus either $(x_1, x_2) \in R_1$ or $(x_1, x_2) \in R_2$, so we have $x_1 \in (\mathcal{P}p_1)^\dagger(R_1)$ or
 245 $x_1 \in (\mathcal{P}p_1)^\dagger(R_2)$, respectively. So, we have $x_1 \in (\mathcal{P}p_1)^\dagger(R_1) \cup (\mathcal{P}p_1)^\dagger(R_2)$.

246 Now, assuming that $x_1 \in (\mathcal{P}p_1)^\dagger(R_1) \cup (\mathcal{P}p_1)^\dagger(R_2)$ either $x_1 \in (\mathcal{P}p_1)^\dagger(R_1)$
 247 or $x_1 \in (\mathcal{P}p_1)^\dagger(R_2)$. Without loss of generality, we can assume $x_1 \in (\mathcal{P}p_1)^\dagger(R_j)$,
 248 where $j \in \{1, 2\}$. Then there exists x_2 that $(x_1, x_2) \in R_j$, then we have $(x_1, x_2) \in$
 249 $R_1 \cup R_2$ that gives $x_1 \in (\mathcal{P}p_1)^\dagger(R_1 \cup R_2)$.

250 **Lemma 2.17.** Assuming that σ_1 and σ_2 are simulation structures of type $R \rightarrow$
 251 $(\mathcal{P}R)^\dagger$, then $\sigma_1 \sqcup \sigma_2$ and $\sigma_1 \sqcap \sigma_2$ are also simulation structures of the same type.

252 *Proof.* Since σ_1 and σ_2 are simulation structures, for every $(x_1, x_2) \in R$, for
 253 $i \in \{1, 2\}$ we have:

$$\alpha(x_1) \subseteq (\mathcal{P}p_1)^\dagger \cdot \sigma_i(x_1, x_2), \quad (6)$$

$$(\mathcal{P}p_2)^\dagger \cdot \sigma_i(x_1, x_2) \subseteq \alpha(x_2). \quad (7)$$

254 First, we prove the case for \sqcup . Since $\alpha(x_1) \subseteq (\mathcal{P}p_1)^\dagger \cdot \sigma_i(x_1, x_2)$ we have the
 255 following:

$$\alpha(x_1) \subseteq (\mathcal{P}p_1)^\dagger \cdot \sigma_1(x_1, x_2) \cup (\mathcal{P}p_1)^\dagger \cdot \sigma_2(x_1, x_2)$$

256 So, by [Lemma 2.16](#) we have $\alpha(x_1) \subseteq (\mathcal{P}p_1)^\dagger(\sigma_1(x_1, x_2) \cup \sigma_2(x_1, x_2))$. Similarly,
 257 we have $(\mathcal{P}p_2)^\dagger \cdot \sigma_i(x_1, x_2) \subseteq \alpha(x_2)$ that gives the following:

$$(\mathcal{P}p_2)^\dagger \cdot \sigma_1(x_1, x_2) \cup (\mathcal{P}p_2)^\dagger \cdot \sigma_2(x_1, x_2) \subseteq \alpha(x_2)$$

258 So, by [Lemma 2.16](#) we have $(\mathcal{P}p_2)^\dagger(\sigma_1(x_1, x_2) \cup \sigma_2(x_1, x_2)) \subseteq \alpha(x_2)$.

259 Now, we prove the case for \sqcap . For \sqcap unlike \sqcup we need to prove that $\sigma_1 \sqcap$
 260 $\sigma_2(x_1, x_2) \in (\mathcal{P}R)^\dagger$. To achieve this, we need to show that assuming π_1, π_2 are
 261 projections of $\sigma_1 \sqcap \sigma_2(x_1, x_2)$, then for $j \in \{1, 2\}$ we have $\pi_j \cdot (\sigma_1 \sqcap \sigma_2)(x_1, x_2) \subseteq$
 262 $\mathcal{P}p_j(R)$. Since $(\mathcal{P}p_j)^\dagger \cdot \sigma_i(x_1, x_2) \subseteq \mathcal{P}p_j(R)$, we have $\pi_j \cdot (\sigma_1 \sqcap \sigma_2)(x_1, x_2) \subseteq$
 263 $\mathcal{P}p_j(R)$, so we have $\sigma_1 \sqcap \sigma_2(x_1, x_2) \in (\mathcal{P}R)^\dagger$, meaning that $\pi_j \cdot (\sigma_1 \sqcap \sigma_2)(x_1, x_2) =$
 264 $(\mathcal{P}p_j)^\dagger \cdot (\sigma_1 \sqcap \sigma_2)(x_1, x_2)$.¹

265 For $j \in \{1, 2\}$ we have

$$\{\text{eq:proj-meet}\} \quad (\mathcal{P}p_j)^\dagger \cdot (\sigma_1 \sqcap \sigma_2(x_1, x_2)) = (\mathcal{P}p_j)^\dagger \cdot \sigma_1(x_1, x_2) \cap (\mathcal{P}p_j)^\dagger \cdot \sigma_2(x_1, x_2). \quad (8)$$

266 Since $\alpha(x_1) \subseteq (\mathcal{P}p_1)^\dagger \cdot \sigma_i(x_1, x_2)$, we have

$$\alpha(x_1) \subseteq (\mathcal{P}p_1)^\dagger \cdot \sigma_1(x_1, x_2) \cap (\mathcal{P}p_1)^\dagger \cdot \sigma_2(x_1, x_2),$$

267 so by (8) we have $\alpha(x_1) \subseteq (\mathcal{P}p_1)^\dagger \cdot (\sigma_1 \sqcap \sigma_2(x_1, x_2))$. Similarly, since $(\mathcal{P}p_2)^\dagger \cdot$
 268 $\sigma_i(x_1, x_2) \subseteq \alpha(x_2)$, we have

$$(\mathcal{P}p_2)^\dagger \cdot \sigma_1(x_1, x_2) \cap (\mathcal{P}p_2)^\dagger \cdot \sigma_2(x_1, x_2) \subseteq \alpha(x_2),$$

so by (8) we have $(\mathcal{P}p_2)^\dagger \cdot (\sigma_1 \sqcap \sigma_2(x_1, x_2)) \subseteq \alpha(x_2)$. \square

`{lem:sim-opsim-inc}`

269 **Lemma 2.18.** *Assuming that $\sigma: R \rightarrow (\mathcal{P}R)^\dagger$ is a simulation structure, and R*
 270 *is symmetric, then for all $(x_1, x_2) \in R$ we have:*

- 271 1. $(\mathcal{P}p_1)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma \cdot s(x_1, x_2) \subseteq (\mathcal{P}p_1)^\dagger \cdot \sigma(x_1, x_2)$
- 272 2. $(\mathcal{P}p_2)^\dagger \cdot \sigma(x_1, x_2) \subseteq (\mathcal{P}p_2)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma \cdot s(x_1, x_2)$

273 *Proof.* We prove the second clause. By (4) for every $(x_1, x_2) \in R$ we have

$$\begin{aligned} \alpha(x_1) &\subseteq (\mathcal{P}p_1)^\dagger \cdot \sigma(x_1, x_2), \\ (\mathcal{P}p_2)^\dagger \cdot \sigma(x_1, x_2) &\subseteq \alpha(x_2). \end{aligned}$$

274 From $\alpha(x_1) \subseteq (\mathcal{P}p_1)^\dagger \cdot \sigma(x_1, x_2)$ since R is symmetric we get $\alpha(x_2) \subseteq (\mathcal{P}p_1)^\dagger \cdot$
 275 $\sigma(x_2, x_1)$, where

$$(\mathcal{P}p_1)^\dagger \cdot \sigma(x_2, x_1) = (\mathcal{P}p_2)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma \cdot s(x_1, x_2).$$

So, from $(\mathcal{P}p_2)^\dagger \cdot \sigma(x_1, x_2) \subseteq \alpha(x_2)$ we have $(\mathcal{P}p_2)^\dagger \cdot \sigma(x_1, x_2) \subseteq (\mathcal{P}p_2)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot$
 $\sigma \cdot s(x_1, x_2)$. Similarly, we can get the other inequation. \square

`{lem:sim-bisim-inc}`

276 **Lemma 2.19.** *Assuming that $\sigma: R \rightarrow (\mathcal{P}R)^\dagger$ is a simulation structure, and*
 277 *$\beta: R \rightarrow (\mathcal{P}R)^\dagger$ is a bisimulation structure,*

¹ PP Note: The last part of the proof is necessary because the type of the codomain of the definition of \sqcap is not $(\mathcal{P}R)^\dagger$, but it is $\mathcal{P}X \times \mathcal{P}X$. Perhaps the epi-mono factorization must be used to cope with this in the abstract case.

278 1. if $\sigma \sqsubseteq \beta$ then we have:

$$\alpha(x_1) = (\mathcal{P}p_1)^\dagger \cdot \sigma(x_1, x_2),$$

279 and if R is symmetric we have

$$(\mathcal{P}p_2)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma \cdot s(x_1, x_2) = \alpha(x_2).$$

280 2. if $\beta \sqsubseteq \sigma$ then we have:

$$(\mathcal{P}p_2)^\dagger \cdot \sigma(x_1, x_2) = \alpha(x_2)$$

281 and if R is symmetric we have

$$\alpha(x_1) = (\mathcal{P}p_1)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma \cdot s(x_1, x_2).$$

282 *Proof.* 1. Since σ is a simulation structure for an arbitrary $(x_1, x_2) \in R$ we
 283 have $\alpha(x_1) \subseteq (\mathcal{P}p_1)^\dagger \cdot \sigma(x_1, x_2)$. Since $\sigma \sqsubseteq \beta$ we have $(\mathcal{P}p_1) \cdot \sigma(x_1, x_2) \subseteq$
 284 $(\mathcal{P}p_1) \cdot \beta(x_1, x_2)$, while $(\mathcal{P}p_1) \cdot \beta(x_1, x_2) = \alpha(x_1)$ by definition of bisimulation.
 285 So we have $\alpha(x_1) = (\mathcal{P}p_1)^\dagger \cdot \sigma(x_1, x_2)$. Then because of the symmetry of R
 286 the second clause is easily achievable by using the equations in (5).
 287 2. This clause can be proven similar to (1). □

288 **Proposition 2.20.** Assuming that $\sigma: R \rightarrow (\mathcal{P}R)^\dagger$ is a simulation structure,
 289 and $\beta: R \rightarrow (\mathcal{P}R)^\dagger$ is a bisimulation structure,

290 1. if $\sigma \sqsubseteq \beta$ then we have:

$$\beta = \sigma \sqcup ((\mathcal{P}s)^\dagger \cdot \sigma \cdot s)$$

291 2. if $\beta \sqsubseteq \sigma$ then we have:

$$\beta = \sigma \cap ((\mathcal{P}s)^\dagger \cdot \sigma \cdot s)$$

292 *Proof.* 1. We need to prove that $\sigma \sqcup ((\mathcal{P}s)^\dagger \cdot \sigma \cdot s)$ is the bisimulation structure.
 293 By Lemma 2.18.(1), for every $(x_1, x_2) \in R$, we have $(\mathcal{P}p_1)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma \cdot s(x_1, x_2) \subseteq$
 294 $(\mathcal{P}p_1)^\dagger \cdot \sigma(x_1, x_2)$, and by Lemma 2.19.(1), we have $(\mathcal{P}p_1)^\dagger \cdot \sigma(x_1, x_2) = \alpha(x_1)$.
 295 So, we have $(\mathcal{P}p_1)^\dagger \cdot \sigma \sqcup (\mathcal{P}p_1)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma \cdot s(x_1, x_2) = \alpha(x_1)$, then by Lemma 2.16
 296 we have $(\mathcal{P}p_1)^\dagger \cdot (\sigma \sqcup ((\mathcal{P}s)^\dagger \cdot \sigma \cdot s))(x_1, x_2) = \alpha(x_1)$.

297 Also, by Lemma 2.19.(1) we have $(\mathcal{P}p_2)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma(x_1, x_2) = \alpha(x_2)$. So,
 298 since we already have $(\mathcal{P}p_2)^\dagger \cdot \sigma(x_1, x_2) \subseteq \alpha(x_2)$ then by Lemma 2.16 we have
 299 $(\mathcal{P}p_2)^\dagger \cdot (\sigma \sqcup ((\mathcal{P}s)^\dagger \cdot \sigma \cdot s))(x_1, x_2) = \alpha(x_2)$.

300 2. We need to prove that $\sigma \cap ((\mathcal{P}s)^\dagger \cdot \sigma \cdot s)$ is the bisimulation structure. For
 301 $i \in \{1, 2\}$, for every $(x_1, x_2) \in R$, we have:

$$\begin{aligned} & (\mathcal{P}p_i)^\dagger \cdot (\sigma \cap ((\mathcal{P}s)^\dagger \cdot \sigma \cdot s))(x_1, x_2) \\ &= (\mathcal{P}p_i)^\dagger \cdot (((\mathcal{P}p_1)^\dagger \cdot \sigma(x_1, x_2) \cap (\mathcal{P}p_1)^\dagger \cdot ((\mathcal{P}s)^\dagger \cdot \sigma \cdot s)(x_1, x_2)) \times ((\mathcal{P}p_2)^\dagger \cdot \sigma(x_1, x_2) \cap (\mathcal{P}p_2)^\dagger \cdot ((\mathcal{P}s)^\dagger \cdot \sigma \cdot s)(x_1, x_2))) \\ &= (\mathcal{P}p_i)^\dagger \cdot \sigma(x_1, x_2) \cap (\mathcal{P}p_i)^\dagger \cdot ((\mathcal{P}s)^\dagger \cdot \sigma \cdot s)(x_1, x_2) \end{aligned}$$

By Lemma 2.18.(1), $(\mathcal{P}p_1)^\dagger \cdot ((\mathcal{P}s)^\dagger \cdot \sigma \cdot s)(x_1, x_2) \subseteq (\mathcal{P}p_1)^\dagger \cdot \sigma(x_1, x_2)$, and by Lemma 2.19.(2) we have $(\mathcal{P}p_1)^\dagger \cdot ((\mathcal{P}s)^\dagger \cdot \sigma \cdot s)(x_1, x_2) = \alpha(x_1)$, so we have $(\mathcal{P}p_1)^\dagger \cdot (\sigma \sqcap ((\mathcal{P}s)^\dagger \cdot \sigma \cdot s))(x_1, x_2) = \alpha(x_1)$.

Also, by Lemma 2.19.(2) we have $(\mathcal{P}p_2)^\dagger \cdot \sigma(x_1, x_2) = \alpha(x_2)$, so, since by Lemma 2.18.(2), we have $(\mathcal{P}p_2)^\dagger \cdot \sigma(x_1, x_2) \subseteq (\mathcal{P}p_2)^\dagger \cdot (\mathcal{P}s)^\dagger \cdot \sigma \cdot s(x_1, x_2)$, so we have $(\mathcal{P}p_2)^\dagger \cdot (\sigma \sqcap ((\mathcal{P}s)^\dagger \cdot \sigma \cdot s))(x_1, x_2) = \alpha(x_2)$. \square

Corollary 2.21. *Assuming that R is a symmetric relation, and $S \neq \emptyset$ is the set of all simulation structures of the type $R \rightarrow (\mathcal{P}R)^\dagger$, then if the bisimulation morphism exists, it is equal with the following morphism:*

$$(\bigsqcup_{\sigma \in S} \sigma) \sqcap (\mathcal{P}s)^\dagger \cdot (\bigsqcup_{\sigma \in S} \sigma) \cdot s$$

Lemma 2.22. *For every $S \in \mathcal{P}R$,*

$$((\mathcal{P}p_1)(S), (\mathcal{P}p_2)(S)) \in (\mathcal{P}R)^\dagger \Leftrightarrow (\mathcal{P}p_1)(S) \subseteq (\mathcal{P}p_1)(R), (\mathcal{P}p_2)(S) \subseteq (\mathcal{P}p_2)(R)$$

{lem:alph-prod}

Lemma 2.23. *Assuming that R is a symmetric relation, and $S \neq \emptyset$ is the set of all simulation structures of the type $R \rightarrow (\mathcal{P}R)^\dagger$, then there exists a simulation structure $\sigma \in S$ that for every (x_1, x_2) , $(\mathcal{P}p_1)^\dagger \cdot \sigma(x_1, x_2) = \alpha(x_1)$.*

Proof. Since $S \neq \emptyset$ there exists $\delta \in S$. We define σ for every (x_1, x_2) as the following:

$$\sigma(x_1, x_2) = (\alpha(x_1), (\mathcal{P}p_2)^\dagger \cdot \delta(x_1, x_2))$$

We have $\sigma(x_1, x_2) \in (\mathcal{P}R)^\dagger$, as $\alpha(x_1) \subseteq \mathcal{P}p_1(R)$ and $(\mathcal{P}p_2)^\dagger \cdot \delta(x_1, x_2) \subseteq \mathcal{P}p_2(R)$ are inherited from δ being a simulation structure. Also, it obviously is a simulation as $(\mathcal{P}p_1)^\dagger \cdot \sigma(x_1, x_2) = \alpha(x_1)$ and $(\mathcal{P}p_2)^\dagger \cdot \sigma(x_1, x_2) \subseteq \alpha(x_2)$ as $(\mathcal{P}p_2)^\dagger \cdot \delta(x_1, x_2) \subseteq \alpha(x_2)$.

{prop:sym-rel-bisim}

Proposition 2.24. *Assuming that R is a symmetric relation, and $S \neq \emptyset$ is the set of all simulation structures of the type $R \rightarrow (\mathcal{P}R)^\dagger$, then the following morphism is the bisimulation structure:*

$$(\bigsqcup_{\sigma \in S} \sigma) \sqcup (\mathcal{P}s)^\dagger \cdot (\bigsqcup_{\sigma \in S} \sigma) \cdot s$$

Proof. For every $(x_1, x_2) \in R$ we have

$$(\mathcal{P}p_1)^\dagger \cdot ((\bigsqcup_{\sigma \in S} \sigma) \sqcup (\mathcal{P}s)^\dagger \cdot (\bigsqcup_{\sigma \in S} \sigma) \cdot s)(x_1, x_2) = (\mathcal{P}p_1)^\dagger \cdot (\bigsqcup_{\sigma \in S} \sigma)(x_1, x_2),$$

and

$$(\mathcal{P}p_2)^\dagger \cdot ((\bigsqcup_{\sigma \in S} \sigma) \sqcup (\mathcal{P}s)^\dagger \cdot (\bigsqcup_{\sigma \in S} \sigma) \cdot s)(x_1, x_2) = (\mathcal{P}p_2)^\dagger \cdot ((\mathcal{P}s)^\dagger \cdot (\bigsqcup_{\sigma \in S} \sigma) \cdot s)(x_1, x_2). \blacksquare$$

By Lemma 2.23 there exists a simulation $\delta \in S$ for which we have $(\mathcal{P}p_1)^\dagger \cdot \delta(x_1, x_2) = \alpha(x_1)$. So, $(\mathcal{P}p_1)^\dagger \cdot (\bigsqcup_{\sigma \in S} \sigma)(x_1, x_2) = \alpha(x_1)$. Then by the equations in (5) we also get $(\mathcal{P}p_2)^\dagger \cdot ((\mathcal{P}s)^\dagger \cdot (\bigsqcup_{\sigma \in S} \sigma) \cdot s)(x_1, x_2) = \alpha(x_2)$. \square

3 Symmetric Simulation in Quantaloids

We generalize [Proposition 2.24](#) in regular quantaloids. A quantaloid is a category enriched with suplattices. Abstractly, first we define an operation that we need on morphisms that takes two simulation witnesses of type $R \rightarrow (FR)^\dagger$ to a morphism of type $R \rightarrow FX \times FX$:

$$\sigma_1 \boxtimes \sigma_2 = (Fp_1)^\dagger \cdot \sigma_1 \sqcap (Fp_1)^\dagger \cdot \sigma_2 \times (Fp_2)^\dagger \cdot \sigma_1 \sqcap (Fp_2)^\dagger \cdot \sigma_2$$

328

{lem:alph-prod-abs}

Lemma 3.1. *Assuming that R is a symmetric relation, and $S \neq \emptyset$ is the set of all simulation witnesses of the type $R \rightarrow (FR)^\dagger$, then there exists a simulation witness $\sigma \in S$ that, $(Fp_1)^\dagger \cdot \sigma = \alpha \cdot p_1$.*

Proof. Since $S \neq \emptyset$ there exists $\delta \in S$. We define σ as the following:

$$\sigma = \langle (\alpha \cdot p_1), (Fp_2)^\dagger \cdot \delta \rangle$$

We have $(Fp_1)^\dagger \cdot \sigma$.

4 Relators

4.1 Two-way similarity in Hughes-Jacobs

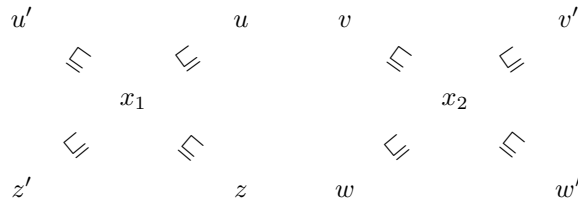
Hughes and Jacobs define two-way similarity as $\sqsubseteq \cap \sqsubseteq^{\text{op}}$. They give a sufficient condition for the two-way similarity to be the bisimilarity. We discuss that this condition does not allow us to say that a symmetric similarity is a bisimilarity. The condition is:

$$\sqsubseteq; (FR_1)^\dagger; \sqsubseteq \cap \sqsubseteq^{\text{op}}; (FR_2)^\dagger; \sqsubseteq^{\text{op}} \subseteq (F(R_1 \cap R_2))^\dagger$$

We need the case that $R_1 = R_2$, and we refer to it with R . So, we need to have

$$\sqsubseteq; (FR)^\dagger; \sqsubseteq \cap \sqsubseteq^{\text{op}}; (FR)^\dagger; \sqsubseteq^{\text{op}} \subseteq (FR)^\dagger.$$

Assuming $(x_1, x_2) \in \sqsubseteq; (FR)^\dagger; \sqsubseteq \cap \sqsubseteq^{\text{op}}; (FR)^\dagger; \sqsubseteq^{\text{op}}$ means that there exist $(u, v), (u', v') \in (FR)^\dagger$, such that $x_1 \sqsubseteq u, u' \sqsubseteq x_1, v \sqsubseteq x_2$, and $x_2 \sqsubseteq v'$. We can also use the symmetry of R , and then from $(x_1, x_2) \in R$ derive that there exist $(w, z), (w', z')$ such that $x_1 \sqsubseteq z, z' \sqsubseteq x_1, w \sqsubseteq x_2$, and $x_2 \sqsubseteq w'$. The situation can be illustrated as the following:



But then how can we derive $(x_1, x_2) \in (FR)^\dagger$?

TODO: Investigate more! Can it be doable really?!

4.2 Uniqueness of the witness in Hughes-Jacobs definition

In this section we set \mathbf{Rel} to be the category that has sets as objects and binary relations as morphisms. We answer the question that why there can be multiple simulation witnesses based on Definition 2.8, while for the same relation, there is only one witness according to Hughes-Jacobs simulation.

Definition 4.1 (Hughes-Jacobs Simulation). For a functor F , and a poset \sqsubseteq over F a HuJ-simulation is a relation r for which there exists a morphism $\sigma: r \rightarrow (Fr)^\dagger$ called *witness* such that the following diagram commutes (\cdot is the relation composition):

$$\begin{array}{ccccc} X & \xleftarrow{p_1} & R & \xrightarrow{p_2} & Y \\ \alpha \downarrow & & \downarrow \sigma & & \downarrow \beta \\ FX & \xleftarrow{Fp_1 \sqsubseteq} & (FR)^\dagger & \xrightarrow{Fp_2 \sqsubseteq} & FY \end{array}$$

At the moment we have limited the discussion to the category of sets and we are talking about the powerset functor. We know that σ is unique in the following diagram:

$$\begin{array}{ccccc} X & \xleftarrow{p_1} & R & \xrightarrow{p_2} & Y \\ \alpha \downarrow & & \downarrow \sigma & & \downarrow \beta \\ \mathcal{P}X & \xleftarrow{\mathcal{P}p_1 \sqsubseteq} & (\mathcal{P}R)^\dagger & \xrightarrow{\mathcal{P}p_2 \sqsubseteq} & \mathcal{P}Y \end{array}$$

It is defined as $\sigma(x_1, x_2) = (\alpha(x_1), \beta(x_2))$. But σ' in the following diagram is not unique:

$$\begin{array}{ccccc} X & \xleftarrow{p_1} & R & \xrightarrow{p_2} & Y \\ \alpha \downarrow & \subseteq & \downarrow \sigma' & \subseteq & \downarrow \beta \\ \mathcal{P}X & \xleftarrow{\mathcal{P}p_1^\dagger} & (\mathcal{P}R)^\dagger & \xrightarrow{\mathcal{P}p_2^\dagger} & \mathcal{P}Y \end{array}$$

Because assuming we have σ , for every given σ' we can define a $\delta: (\mathcal{P}R)^\dagger \rightarrow \subseteq; (\mathcal{P}R)^\dagger; \subseteq$ that $\sigma = \delta \cdot \sigma'$, i.e., the following diagram commutes:

$$\begin{array}{ccccc} X & \xleftarrow{p_1} & R & \xrightarrow{p_2} & Y \\ \alpha \downarrow & & \downarrow \sigma' & & \downarrow \beta \\ \mathcal{P}X & & (\mathcal{P}R)^\dagger & & \mathcal{P}Y \\ id \downarrow & & \downarrow \delta & & \downarrow id \\ \mathcal{P}X & \xleftarrow{\mathcal{P}p_1 \sqsubseteq} & (\mathcal{P}R)^\dagger & \xrightarrow{\mathcal{P}p_2 \sqsubseteq} & \mathcal{P}Y \end{array}$$

To define δ , we define $c: (\mathcal{P}R)^\dagger \rightarrow ((\mathcal{P}R)^\dagger \times R) + (\mathcal{P}R)^\dagger$ and $u: ((\mathcal{P}R)^\dagger \times R) + (\mathcal{P}R)^\dagger \rightarrow \subseteq; (\mathcal{P}R)^\dagger; \subseteq$ and then we define $\delta = u \cdot c$. Here are the definitions for c

and u :

$$c(w) = \begin{cases} \text{inl}(w, (x_1, x_2)) & \exists x_1, x_2, \sigma'(x_1, x_2) = w \\ \text{inr } w & \text{o.w} \end{cases}$$

$$u(\text{inl } w, (x_1, x_2)) = (\alpha(x_1), \alpha(x_2))$$

$$u(\text{inr } w) = w$$

4.3 Symmetric simulation

Notation 4.2. From now on, we show relations with small letters, and for two relations r_1 and r_2 by $r_1 \leq r_2$ we mean $r_1 \subseteq r_2$. Also, we show the category of relations over set that we represent by spans with **Span**, and **Rel** is the category of sets and binary relations between them.

Lemma 4.3. $r: X \rightarrowtail Y$ is a morphism in **Rel** iff there is an object (r, p_1, p_2) in **Span**. {lem:rel-span-equiv}

Proof. (\Rightarrow): $r: X \rightarrowtail Y$ being a morphism in **Rel** means that in **Set** there exist an object r with a unique mono of type $r \rightarrow X \times Y$ that is a pairing that we show with $\langle p_1, p_2 \rangle$. So, (r, p_1, p_2) form an object in **Span**.

(\Leftarrow): If (r, p_1, p_2) is an object in **Span**, then r is a binary relation from X to Y so, it is a morphism of type $X \rightarrowtail Y$ in **Rel**. \square

The above translation seems to be true in a more general case, where **Span** and **Rel** are defined on an arbitrary category (the latter is called an allegory then).

Definition 4.4 (Relator). Assuming F is a functor on **Set**, a F -relator or simply a relator **R** is a monotone map that sends a morphism of **Rel** that is a relation $X \rightarrowtail Y$ to $FX \rightarrowtail FY$.

Definition 4.5 (Hermida-Jacobs Simulation). For a relator **R** on a functor F a HJ-simulation from a coalgebra $\alpha: X \rightarrow FX$ to a coalgebra $\beta: Y \rightarrow FY$ is a relation r for which there exists a morphism $\sigma: r \rightarrow \mathbf{R}r$ called *witness* such that the following diagram commutes ($;$ is the relation composition): {def:hej-sim}

$$\begin{array}{ccccc} X & \xleftarrow{p_1} & r & \xrightarrow{p_2} & Y \\ \alpha \downarrow & & \downarrow \sigma & & \downarrow \beta \\ FX & \xleftarrow{(Fp_1)\mathbf{R}} & \mathbf{R}r & \xrightarrow{(Fp_2)\mathbf{R}} & FY \end{array} \quad (9) \quad \{\text{eq:hej-sim}\}$$

Definition 4.6 (Relator-based Simulation). Given a relator **R**, a relation $r: X \rightarrowtail Y$ is a **R**-simulation from a coalgebra $\alpha: X \rightarrow FX$ to a coalgebra $\beta: Y \rightarrow FY$ if $r \leq \alpha; \mathbf{R}r; \beta^{\text{op}}$, i.e, if $(x, y) \in r$ entails $(\alpha(x), \beta(y)) \in \mathbf{R}r$, for all $x \in X$ and $y \in Y$.

Definition 4.7 (Symmetric Relator). A relator **R** is symmetric if and only if for every relation r we have $\mathbf{R}(r^{\text{op}}) = (\mathbf{R}r)^{\text{op}}$.

Definition 4.8 (Relator-based Bisimulation). Given a relator \mathbf{R} , a relation $r: X \rightarrowtail Y$ is a \mathbf{R} -bisimulation from a coalgebra $\alpha: X \rightarrow FX$ to a coalgebra $\beta: Y \rightarrow FY$ if r is a \mathbf{R} -simulation, and \mathbf{R} is a symmetric relator.

Notation 4.9. From now on we refer to relator-based simulations of a relator \mathbf{R} with \mathbf{R} -simulation. If we talk about a HJ-simulation we specify the witness.

{lem:sim-simp}

Lemma 4.10. r being a \mathbf{R} -simulation means that assuming $x \ r \ y$ we have $\alpha(x) \ \mathbf{R}r \ \beta(y)$.

Proof. r being a \mathbf{R} -simulation means $r \leq \beta^{\text{op}} \cdot \mathbf{R}r \cdot \alpha$, meaning that if $x \ r \ y$ then $x \ \alpha; \mathbf{R}r; \beta^{\text{op}} \ y$, and it means that there exist $w \in \mathbf{R}r$ that its first element is equal with $\alpha(x)$ and its second element is equal with $\beta(y)$, enabling us to say $\alpha(x) \ \mathbf{R}r \ \beta(y)$. \square

Proposition 4.11. For a relator \mathbf{R} of a functor F , a relation r is a \mathbf{R} -simulation from $\alpha: X \rightarrow FX$ to $\beta: Y \rightarrow FY$ iff it is a HJ-simulation with the witness $\sigma: r \rightarrow \mathbf{R}r$.

Proof. (\Rightarrow) : r being a \mathbf{R} -simulation means that $x \ r \ y$ gives $\alpha(x) \ \mathbf{R}r \ \beta(y)$. Since r and $\mathbf{R}r$ are both relations, by Lemma 4.3 there exist objects (r, p_1, p_2) and $(\mathbf{R}r, (Fp_1)^{\mathbf{R}}, (Fp_2)^{\mathbf{R}})$ in **Span**. We define $\sigma: r \rightarrow \mathbf{R}r$ to be $\sigma(x, y) = (\alpha(x), \beta(y))$. σ commutes in (9), so we have a HJ-simulation.

(\Leftarrow) : Assuming we have a σ that commutes in (9), we want to prove that if $x \ r \ y$ we have $\alpha(x) \ \mathbf{R}r \ \beta(y)$. By (9) we have $\alpha \cdot p_1 = (Fp_1)^{\mathbf{R}} \cdot \sigma$ and $\alpha \cdot p_2 = (Fp_2)^{\mathbf{R}} \cdot \sigma$. It means that since $x \ r \ y$ then exists $w \in \mathbf{R}r$, such that $\sigma(x, y) = w$, where $(Fp_1)^{\mathbf{R}}(w) = \alpha(x)$ and $(Fp_2)^{\mathbf{R}}(w) = \beta(y)$. So, we have $\alpha(x) \ \mathbf{R}r \ \beta(y)$. \square

Proposition 4.12. For an arbitrary relator \mathbf{R} on a functor F , if a relation r is a HJ-simulation, the witness is unique.

Proof. It only relies on the fact that $(Fp_1)^{\mathbf{R}}$ and $(Fp_2)^{\mathbf{R}}$ in (4.5) are jointly monic. \square

Definition 4.13. We call $\hat{\mathbf{R}}$ a symmetrization of a relator \mathbf{R} iff for a relation r it is defined as follows:

$$\hat{\mathbf{R}}r = \mathbf{R}r \cap (\mathbf{R}(r^{\text{op}}))^{\text{op}}$$

Proposition 4.14. For every relator \mathbf{R} , $\hat{\mathbf{R}}$ is a relator.

Proof. Almost obvious! \square

Proposition 4.15. Assuming that \mathbf{R} is a relator, and r and r^{op} are both \mathbf{R} -simulations from a coalgebra $\alpha: X \rightarrow FX$ to a coalgebra $\beta: Y \rightarrow FY$ and vice-versa respectively, then r is also a $\hat{\mathbf{R}}$ -simulation.

Proof. We need to prove that $x \ r \ y$ gives $\alpha(x) \ \hat{\mathbf{R}}r \ \beta(y)$. r being a \mathbf{R} -simulation means that assuming $x \ r \ y$ we have $\alpha(x) \ \mathbf{R}r \ \beta(y)$. So, we are left to prove $\alpha(x) \ (\mathbf{R}(r^{\text{op}}))^{\text{op}} \ \beta(y)$. $x \ r \ y$ gives $y \ r^{\text{op}} \ x$, and r^{op} being a \mathbf{R} -simulation from β to α gives $\beta(y) \ \mathbf{R}r^{\text{op}} \ \alpha(x)$. So, we have $\alpha(x) \ (\mathbf{R}(r^{\text{op}}))^{\text{op}} \ \beta(y)$. \square

414 **Corollary 4.16.** *Assuming that \mathbf{R} is a relator, and r is a symmetric \mathbf{R} -*
 415 *simulation from a coalgebra $\alpha: X \rightarrow FX$ to itself, then r is also a $\hat{\mathbf{R}}$ -simulation.*

416 **Remark 4.17.** If we want to have the previous corollary for two different coal-
 417 gebras $\alpha: X \rightarrow FX$ and $\beta: X \rightarrow FX$, we need to assume that r^{op} is also a
 418 \mathbf{R} -simulation from β to α .

419 **Proposition 4.18.** *$\hat{\mathbf{R}}$ is a symmetric relator, i.e., every $\hat{\mathbf{R}}$ -simulation is actu-*
 420 *ally a $\hat{\mathbf{R}}$ -bisimulation.*

Proof.

$$\begin{aligned}
 \hat{\mathbf{R}}(r^{\text{op}}) &= \mathbf{R}(r^{\text{op}}) \cap (\mathbf{R}(r^{\text{op}})^{\text{op}})^{\text{op}} \\
 &= \mathbf{R}(r^{\text{op}}) \cap (\mathbf{R}r)^{\text{op}} \\
 &= (\mathbf{R}r)^{\text{op}} \cap \mathbf{R}(r^{\text{op}}) \\
 &= (((\mathbf{R}r)^{\text{op}} \cap \mathbf{R}(r^{\text{op}}))^{\text{op}})^{\text{op}} \\
 &= (\mathbf{R}r \cap (\mathbf{R}(r^{\text{op}})^{\text{op}})^{\text{op}})^{\text{op}} \\
 &= (\hat{\mathbf{R}}r)^{\text{op}}
 \end{aligned}$$

□

421 **Proposition 4.19.** *Assuming that \mathbf{R} is a symmetric relator, and r is a \mathbf{R} -*
 422 *simulation from a coalgebra (X, α) to itself, then r_s^{op} is a \mathbf{R} -simulation as well.*

Proof. It is easy to directly show that r^{op} is a \mathbf{R} -simulation. □

423 **Corollary 4.20.** *Assuming that \mathbf{R} is a symmetric relator, then the \mathbf{R} -*
 424 *similarity from a coalgebra (X, α) to itself is a symmetric relation.*

425 **Proposition 4.21.** *Assuming that \mathbf{R} is a symmetric relator, then for every r*
 426 *that is a \mathbf{R} -simulation, r^{op} is also a \mathbf{R} -simulation.*

Proof. Assuming $y \ r^{\text{op}} x$ we have $x \ r \ y$. Since r is \mathbf{R} -simulation we have
 $\alpha(x) \ \mathbf{R}r \ \beta(y)$. So, we have $\beta(y) \ (\mathbf{R}r)^{\text{op}} \ \alpha(x)$, and since \mathbf{R} is symmetric we
 have $\beta(y) \ \mathbf{R}r^{\text{op}} \ \alpha(x)$. □

427 **Definition 4.22 (Behavioural Equivalence).** Two states x and y of two coal-
 428 gebras (X, α) and (Y, β) are behaviourally equivalent iff there exist a coalgebra
 429 (Z, γ) and coalgebra morphisms $f: (X, \alpha) \rightarrow (Z, \gamma)$ and $g: (Y, \beta) \rightarrow (Z, \gamma)$ such
 430 that $f(x) = g(y)$. The relation r consisting of all behaviourally equivalent states
 431 of these two coalgebras is called behavioural equivalence.

432 **Definition 4.23 (Difunctional Relation).** A relation $r: X \rightarrowtail Y$ is difunc-
 433 tional iff there are functions $f: X \rightarrow Z$ and $g: Y \rightarrow Z$ such that for every
 434 $(x, y) \in R$ we have $f(x) = g(y)$.

Definition 4.24 (Soundness and Completeness of \mathbf{R} -similarity). For a
 relator \mathbf{R} the \mathbf{R} -similarity from a coalgebra (X, α) to a coalgebra (Y, β) is sound
 iff it is less than or equal to their behavioural equivalence, and it is complete iff
 it is greater than or equal to their behavioural equivalence.

Theorem 4.25. *Let \mathbf{R} be a relator for a functor F :*

1. *If for all functions $f: X \rightarrow A$ and $g: Y \rightarrow A$, $\mathbf{R}(g^{\text{op}} \cdot f) \geq (Fg)^{\text{op}} \cdot Ff$, then \mathbf{R} -similarity is complete.*
2. *If \mathbf{R} preserves difunctional relations and for every epi-cospan $(f: X \rightarrow A, g: Y \rightarrow A) \in \mathbf{Set}$, $\mathbf{R}(g^{\text{op}} \cdot f) \leq (Fg)^{\text{op}} \cdot Ff$, then \mathbf{R} -similarity is sound.*

□

{prop:difunc-preser}

Proposition 4.26. *Assuming that \mathbf{R} is a F -relator that preserves difunctional relations, then $\hat{\mathbf{R}}$ does the same.*

Proof. Assuming r is difunctional, then there exist $f_1, f_2: FX \rightarrow FZ$ and $g_1, g_2: FY \rightarrow FZ$ such that $p \mathbf{R} r q$ iff $f_1(p) = g_1(q)$ and $q \mathbf{R} r^{\text{op}} p$ iff $f_2(p) = g_2(q)$. By the definition of $\hat{\mathbf{R}}$ we have $p \hat{\mathbf{R}} r q$ iff $p \mathbf{R} r q$ and $p (\mathbf{R} r^{\text{op}})^{\text{op}} q$ that is equivalent to say that $\langle f_1, f_2 \rangle(p) = \langle g_1, g_2 \rangle(q)$. □

Corollary 4.27. *Assuming that \mathbf{R} is a F -relator that preserves difunctional relations, for every symmetric relation r we have $\hat{\mathbf{R}}r = \mathbf{R}r$.*

Proof. \mathbf{R} -similarity being symmetric means that for the f_1, f_2, g_1 and g_2 in the proof of Proposition 4.26, we have $f_1 = f_2$ and $g_1 = g_2$. □

Assuming that \mathbf{R} -similarity is complete, does not guarantee that $\hat{\mathbf{R}}$ -similarity is sound and complete. We give a counter-example. Assuming that \mathbf{R} is a \mathcal{P} -relator that takes $r: X \rightarrowtail Y$ to $\mathcal{P}X \times \mathcal{P}Y$, then $\hat{\mathbf{R}}r = \mathcal{P}X \times \mathcal{P}Y$ as well. It means that for every coalgebras $(X, \alpha), (Y, \beta)$, $\hat{\mathbf{R}}$ -similarity is equal to $X \times Y$, which is rare to be equal to behavioural equivalence. For example, if we take $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, y_3\}$, and we define α and β as

$$\alpha(x) = \begin{cases} \{x_1, x_2\} & x = x_1 \\ \{x_2\} & x = x_2 \end{cases}$$

and

$$\beta(y) = \begin{cases} \{y_3\} & y = y_1 \\ Y & y = y_2 \\ \emptyset & y = y_3 \end{cases}$$

then at least (x_1, y_3) is not in the behavioural equivalence, while it is in $\hat{\mathbf{R}}$ -similarity.

Proposition 4.28. *Assuming that \mathbf{R} -similarity is symmetric and complete, then $\hat{\mathbf{R}}$ -similarity from a coalgebra $\alpha: X \rightarrow FX$ to itself is sound and complete.*

459 *Proof.* (Completeness): We show \mathbf{R} -similarity with r_s and $\hat{\mathbf{R}}$ -similarity with $r_{\hat{s}}$.
 460 Also, we show the behavioural equivalence with r_b . Since

461 **Proposition 4.29.** *Assuming that \mathbf{R} is a F -relator (F is a set functor), that*
 462 *for every functions $f: X \rightarrow Z$ and $g: Y \rightarrow Z$, we have $\mathbf{R}(g^{\text{op}} \cdot f) \geq (Fg)^{\text{op}} \cdot Ff$,*
 463 *then $\hat{\mathbf{R}}$ -similarity is complete.*

Proof. We need to prove that for every functions $f: X \rightarrow Z$ and $g: Y \rightarrow Z$ we have $\hat{\mathbf{R}}(g^{\text{op}} \cdot f) \geq (Fg)^{\text{op}} \cdot Ff$. By the assumption we have $\mathbf{R}(g^{\text{op}} \cdot f) \geq (Fg)^{\text{op}} \cdot Ff$. Also, again from the assumption we have $\mathbf{R}(f^{\text{op}} \cdot g) \geq (Ff)^{\text{op}} \cdot Fg$ that gives $(\mathbf{R}(f^{\text{op}} \cdot g))^{\text{op}} \geq (Fg)^{\text{op}} \cdot Ff$. So, we have $\hat{\mathbf{R}}(g^{\text{op}} \cdot f) \geq (Fg)^{\text{op}} \cdot Ff$. \square

464 **Proposition 4.30.** *Assuming that \mathbf{R} is a symmetric relator for a functor*
 465 *$F: \mathbf{Set} \rightarrow \mathbf{Set}$, then the \mathbf{R} -bisimilarity from a coalgebra $\alpha: X \rightarrow FX$ to it-*
 466 *self is sound, using the axiom of choice.*

467 *Proof.* We call the bisimilarity relation r , and we assume $x_1 r x_2$, now we need
 468 to prove that x_1 and x_2 are behaviourally equivalent. We take $Z = X/r$, where
 469 $X/r = \{[x] \mid [x] = \{y \mid x r y\}\}$. Now, we define the coalgebra homomorphism
 470 $f: X \rightarrow X/r$ as $f(x) = [x]$. So, assuming $x_1 r x_2$ gives $f(x_1) = f(x_2)$. Now,
 471 assuming that exists a choice function $c: X/r \rightarrow X$ that $c \cdot f = \text{id}_X$, we define
 472 $\gamma: X/r \rightarrow F(X/r)$, as $\gamma([x]) = Ff \cdot \alpha \cdot c([x])$. Now, we have

$$\begin{aligned} \gamma \cdot f &= Ff \cdot \alpha \cdot c \cdot f \\ &= Ff \cdot \alpha. \end{aligned}$$

So, x_1 and x_2 are behaviourally equivalent. So, the \mathbf{R} -bisimilarity is sound. \square

473 **Corollary 4.31.** *Assuming that a relator \mathbf{R} over a functor $F: \mathbf{Set} \rightarrow \mathbf{Set}$ satis-*
 474 *fies $\mathbf{R}(g^{\text{op}} \cdot f) \geq (Fg)^{\text{op}} \cdot Ff$ for every functions $f: X \rightarrow Z$ and $g: Y \rightarrow Z$, then*
 475 *$\hat{\mathbf{R}}$ -bisimilarity from a coalgebra $\alpha: X \rightarrow FX$ to itself is sound and complete,*
 476 *using the axiom of choice.*

477 4.4 Egli-Milner relator and Barr relators

478 **Definition 4.32.** We call the map $\mathbf{L}: \mathbf{Rel} \rightarrow \mathbf{Rel}$ the Egli-Milner \mathcal{P} -relator,
 479 whenever for every relation $r: X \rightarrowtail Y$ it is defined as follows:

$$\mathbf{L}r = \{(S, T) \mid x \in S \Rightarrow \exists y \in T, x r y\}$$

480 Egli-Milner relator is not sound or complete, although its symmetrization is
 481 sound and complete.

482 **Proposition 4.33.** *$\hat{\mathbf{L}}$ -similarity from a coalgebra (α, X) to (β, Y) is sound and*
 483 *complete.*

484 *Proof.* We need to prove that for every functions $f: X \rightarrow Z$ and $g: Y \rightarrow Z$,
 485 $\hat{\mathbf{L}}(g^{\text{op}} \cdot f) = (\mathcal{P}g)^{\text{op}} \cdot \mathcal{P}f$. We have $S \hat{\mathbf{L}}(g^{\text{op}} \cdot f) T$ iff $S \mathbf{L}(g^{\text{op}} \cdot f) T$ and $T \mathbf{L}(f^{\text{op}} \cdot g) S$.
 486 Then we have

$$\begin{aligned} S \mathbf{L}(g^{\text{op}} \cdot f) T & \\ \iff \forall x \in S, \exists y \in T, x \cdot g^{\text{op}} \cdot f \cdot y & \\ \iff \forall x \in S, \exists y \in T, z \in Z, x \cdot f \cdot z, y \cdot g \cdot z, & \end{aligned}$$

487 and

$$\begin{aligned} T \mathbf{L}(f^{\text{op}} \cdot g) S & \\ \iff \forall y \in T, \exists x \in S, y \cdot f^{\text{op}} \cdot g \cdot x & \\ \iff \forall y \in T, \exists x \in S, z \in Z, x \cdot f \cdot z, y \cdot g \cdot z. & \end{aligned}$$

488 It is equivalent with the following:

$$\begin{aligned} \forall x \in S, \exists y \in T, f(x) = g(y), \\ \forall y \in T, \exists x \in S, f(x) = g(y). \end{aligned}$$

489 Equivalently, $\text{Im}(f|_S) = \text{Im}(g|_T)$, and we call images U that is in $\mathcal{P}Z$. So, we
 490 equivalently have

$$\begin{aligned} S \mathcal{P}f U, T \mathcal{P}g U & \\ \iff S \mathcal{P}f U, U (\mathcal{P}g)^{\text{op}} T & \\ \iff S (\mathcal{P}g)^{\text{op}} \cdot \mathcal{P}f T & \end{aligned}$$

□

491 For every relation $r \rightarrow X \rightarrow Y$ $\mathbf{L}r \subseteq \mathbf{L}r = \mathbf{L}r \subseteq \subseteq \mathbf{L}r; \subseteq$.

492 Barr relator is a generalization of the Egli-Milner relator, where the functor
 493 is generalized.

494 **Definition 4.34 (Barr relator).** A relator over a functor F is a Barr relator,
 495 shown by \bar{F} , iff for a relation $r: X \rightarrow Y$, and a span $(\pi_1: A \rightarrow X, \pi_2: A \rightarrow Y)$
 496 that $r = \pi_2 \cdot \pi_1^{\text{op}}$ we have:

$$\bar{F}r = F\pi_2 \cdot (F\pi_1)^{\text{op}}$$

497 **Proposition 4.35.** For every set-functor F , the barr relator \bar{F} is symmetric.

498 *Proof.* Assuming that for a span $(\pi_1: A \rightarrow X, \pi_2: A \rightarrow Y)$ we have $r = \pi_2 \cdot \pi_1^{\text{op}}$,
 499 then we have:

$$\begin{aligned} \hat{\bar{F}}r & \\ = \bar{F}r \cap (\bar{F}r)^{\text{op}} & \\ = \bar{F}(\pi_2 \cdot \pi_1^{\text{op}}) \cap (\bar{F}(\pi_2 \cdot \pi_1^{\text{op}}))^{\text{op}} & \\ = \bar{F}(\pi_2 \cdot \pi_1^{\text{op}}) \cap (\bar{F}(\pi_1 \cdot \pi_2^{\text{op}}))^{\text{op}} & \end{aligned}$$

$$\begin{aligned}
&= F\pi_2 \cdot (F\pi_1)^{\text{op}} \cap (F\pi_1 \cdot (F\pi_2)^{\text{op}})^{\text{op}} \\
&= F\pi_2 \cdot (F\pi_1)^{\text{op}} \cap F\pi_2 \cdot (F\pi_1)^{\text{op}} \\
&= F\pi_2 \cdot (F\pi_1)^{\text{op}} \\
&= \bar{F}r
\end{aligned}$$

□

500 **Proposition 4.36.** \hat{L} is a Barr relator.

501 *Proof.* **TODO: Finish.**

502 **Definition 4.37 (Mid-lax Barr relator).** Given a relation r , and take a span
 503 $(\pi_1: A \rightarrow X, \pi_2: A \rightarrow Y)$ that $r = \pi_2 \cdot \pi_1^{\text{op}}$. Assuming that \sqsubseteq is a partial order
 504 over a functor F , then the relator over F and shown with \vec{F} is a *mid-lax Barr*
 505 *relator* if we have:

$$\vec{F}r = F\pi_2 \cdot \sqsubseteq \cdot (F\pi_1)^{\text{op}}$$

506 **Proposition 4.38.** For every functor $F: \mathbf{Set} \rightarrow \mathbf{Set}$, the symmetrization of the
 507 *mid-lax Bar relator* is equal with the *Barr relator*.

508 *Proof.* **TODO: Finish.**

509 **Proposition 4.39.** For a span $(\pi_1: A \rightarrow X, \pi_2: A \rightarrow Y)$ the following proposi-
 510 tions hold:

$$\begin{aligned}
511 \quad 1. \mathcal{P}\pi_2 \cdot \subseteq \cdot (\mathcal{P}\pi_1)^{\text{op}} &= \subseteq \cdot \mathcal{P}\pi_2 \cdot (\mathcal{P}\pi_1)^{\text{op}} \\
512 \quad 2. \mathcal{P}\pi_1 \cdot \subseteq \cdot (\mathcal{P}\pi_2)^{\text{op}} &= \subseteq \cdot \mathcal{P}\pi_1 \cdot (\mathcal{P}\pi_2)^{\text{op}}
\end{aligned}$$

513 *Proof.* Without loss of generality, we assume $i, j \in \{1, 2\}$, and $i \neq j$, and prove
 514 $\mathcal{P}\pi_j \cdot \subseteq \cdot (\mathcal{P}\pi_i)^{\text{op}} = \subseteq \cdot \mathcal{P}\pi_j \cdot (\mathcal{P}\pi_i)^{\text{op}}$.

515 Assuming $x \mathcal{P}\pi_j \cdot \subseteq \cdot (\mathcal{P}\pi_i)^{\text{op}} y$, then exist z and z' such that

$$\begin{aligned}
&z (\mathcal{P}\pi_i) x, \\
&z \subseteq z', \\
&z' (\mathcal{P}\pi_j) y.
\end{aligned}$$

516 Then from $z \subseteq z'$ we get $\mathcal{P}\pi_j(z) \subseteq y$. So, we have $z \subseteq \mathcal{P}\pi_j y$, thus
 517 $x \subseteq \mathcal{P}\pi_j \cdot (\mathcal{P}\pi_i)^{\text{op}} y$.

518 Now, assuming $x \subseteq \mathcal{P}\pi_j \cdot (\mathcal{P}\pi_i)^{\text{op}} y$, then there exist z and y' such that

$$\begin{aligned}
&z (\mathcal{P}\pi_i) x, \\
&z (\mathcal{P}\pi_j) y', \\
&y' \subseteq y.
\end{aligned}$$

We take the set $w = z \cup (\mathcal{P}\pi_i(z) \times y)$ for which we have $z \subseteq w$ and $\mathcal{P}\pi_j(w) = y$.
 So, we have $w (\mathcal{P}\pi_j) y$, $z \subseteq w$, and $z (\mathcal{P}\pi_i) x$ that gives $x \mathcal{P}\pi_j \cdot \subseteq \cdot (\mathcal{P}\pi_i)^{\text{op}} y$. □

Proposition 4.40. *For a span $(\pi_1: A \rightarrow X, \pi_2: A \rightarrow Y)$, assuming that F is either the maybe monad, the subdistribution monad or $FX = \mathcal{P}(X \times A)$, where A is a set of labels, then the following propositions hold:*

1. $F\pi_2 \cdot \sqsubseteq \cdot (F\pi_1)^{\text{op}} = \sqsubseteq \cdot F\pi_2 \cdot (F\pi_1)^{\text{op}}$
2. $F\pi_1 \cdot \sqsubseteq \cdot (F\pi_2)^{\text{op}} = \sqsubseteq \cdot F\pi_1 \cdot (F\pi_2)^{\text{op}}$

Proof. **TODO: Finish.**

Definition 4.41 (Natural Order Structure). A *natural order structure* on a functor F is a preorder \sqsubseteq on each Hom -set of the form $\text{Hom}(X, FY)$ such that if $\alpha \sqsubseteq \beta$ in $\text{Hom}(X, FY)$, $f: X \rightarrow X'$, $g: Y \rightarrow Y'$, then:

- (I) $\alpha \cdot f \sqsubseteq \beta \cdot f$ in $\text{Hom}(X', Y)$.
- (II) $Fg \cdot \alpha \sqsubseteq Fg \cdot \beta$ in $\text{Hom}(X, Y')$.

Definition 4.42 (Good Order Structure). A *good order structure* on a functor F is a preorder \sqsubseteq on each Hom -set of the form $\text{Hom}(X, FY)$ that is a natural order structure, and if $h: X \rightarrow FZ$, $k: X \rightarrow FY$, $g: Y \rightarrow Z$, $h \sqsubseteq Fg \cdot k$ in $\text{Hom}(X, FZ)$, then there is $k': X \rightarrow FY$ such that $k' \sqsubseteq k$ in $\text{Hom}(X, FY)$ and $h = Fg \cdot k'$.

Definition 4.43 (Cogood Order Structure). A *cogood order structure* on a functor F is a preorder \sqsubseteq on each Hom -set of the form $\text{Hom}(X, FY)$ that is a natural order structure, and if $h: X \rightarrow FZ$, $k: X \rightarrow FY$, $g: Y \rightarrow Z$, $Fg \cdot k \sqsubseteq h$ in $\text{Hom}(X, FZ)$, then there is $k': X \rightarrow FY$ such that $k \sqsubseteq k'$ in $\text{Hom}(X, FY)$ and $h = Fg \cdot k'$.

Lemma 4.44. *Assuming that a functor F has an order structure \sqsubseteq that is good, then for every $f \in \text{Hom}(X, FY)$ we have:*

- (I) $Ff \cdot \sqsupseteq = \sqsupseteq \cdot Ff$
- (II) $(Ff)^{\text{op}} \cdot \sqsubseteq = \sqsubseteq \cdot (Ff)^{\text{op}}$

Proof. (I) Assuming $t \cdot Ff \cdot \sqsupseteq x$, there exists s such that $t \sqsupseteq s$ and $Ff(s) = x$. Since \sqsubseteq is good, and thus natural, by [Definition 4.41](#), from $s \sqsubseteq t$ we get $x \sqsubseteq Ff(t)$ that is $t \sqsupseteq \cdot Ff x$.

Assuming $t \sqsupseteq \cdot Ff x$, there exists y such that $Ff(t) = y$ and $y \sqsupseteq x$. By [Definition 4.42](#) since $Ff(t) \sqsupseteq x$ there exists s that $t \sqsupseteq s$ and $Ff(s) = x$ that is $t \cdot Ff \cdot \sqsubseteq s$.

(II) Basically, by definition of op and relation composition we have

$$\begin{aligned} (Ff \cdot \sqsubseteq)^{\text{op}} &= \sqsupseteq \cdot (Ff)^{\text{op}}, \\ (\sqsubseteq \cdot Ff)^{\text{op}} &= (Ff)^{\text{op}} \cdot \sqsupseteq. \end{aligned}$$

So it follows directly from applying op on both sides of (I). \square

Lemma 4.45. *Assuming that a functor F has an order structure \sqsubseteq that is cogood, then for every $f \in \text{Hom}(X, FY)$ we have:*

$$\begin{aligned}
553 \quad (I) \quad Ff \cdot \sqsubseteq &= \sqsubseteq \cdot Ff \\
554 \quad (II) \quad (Ff)^{\text{op}} \cdot \sqsupseteq &= \sqsupseteq \cdot (Ff)^{\text{op}}
\end{aligned}$$

555 *Proof.* (I) Assuming $t Ff \cdot \sqsubseteq x$, there exists s such that $t \sqsubseteq s$ and $Ff(s) = x$.
 556 Since \sqsubseteq is cogood, and thus natural, by [Definition 4.41](#), from $t \sqsubseteq s$ we get
 557 $Ff(t) \sqsubseteq x$ that is $t \sqsubseteq \cdot Ff x$.

558 Assuming $t \sqsubseteq \cdot Ff x$, there exists y such that $Ff(t) = y$ and $y \sqsubseteq x$. By [Def-](#)
 559 [inition 4.43](#) since $Ff(t) \sqsubseteq x$ there exists s that $t \sqsubseteq s$ and $Ff(s) = x$ that is
 560 $t Ff \cdot \sqsubseteq s$.

561 (II) Basically, by definition of op and relation composition we have

$$\begin{aligned}
(Ff \cdot \sqsubseteq)^{\text{op}} &= \sqsupseteq \cdot (Ff)^{\text{op}}, \\
(\sqsubseteq \cdot Ff)^{\text{op}} &= (Ff)^{\text{op}} \cdot \sqsupseteq.
\end{aligned}$$

So it follows directly from applying op on both sides of (I). \square

562 **Proposition 4.46.** *For a span $(\pi_1: A \rightarrow X, \pi_2: A \rightarrow Y)$, assuming that F has*
 563 *an order structure \sqsubseteq , the following propositions hold:*

- 564 1. *If \sqsubseteq is good, we have $F\pi_2 \cdot \sqsubseteq \cdot (F\pi_1)^{\text{op}} = F\pi_2 \cdot (F\pi_1)^{\text{op}} \cdot \sqsubseteq$*
- 565 2. *If \sqsubseteq is cogood, we have $F\pi_2 \cdot \sqsubseteq \cdot (F\pi_1)^{\text{op}} = \sqsubseteq \cdot F\pi_2 \cdot (F\pi_1)^{\text{op}}$*
- 566 3. *If \sqsubseteq is both good and cogood, all the following are equal:*
 - 567 • $F\pi_2 \cdot \sqsubseteq \cdot (F\pi_1)^{\text{op}}$
 - 568 • $F\pi_2 \cdot (F\pi_1)^{\text{op}} \cdot \sqsubseteq$
 - 569 • $\sqsubseteq \cdot F\pi_2 \cdot (F\pi_1)^{\text{op}}$
 - 570 • $\sqsubseteq \cdot F\pi_2 \cdot (F\pi_1)^{\text{op}} \cdot \sqsubseteq$

Proof. They all follow in an obvious way from [Lemma 4.44](#) and [Lemma 4.45](#).
 The last one needs $\sqsubseteq \cdot \sqsubseteq = \sqsubseteq$ that comes from transitivity of \sqsubseteq . \square

571 **Example 4.47.** In the category of sets, subset over the powerset functor is an
 572 example of a good order. For every $h: X \rightarrow \mathcal{P}Z$, $k: X \rightarrow \mathcal{P}Y$, and $g: Y \rightarrow Z$,
 573 such that $h \subseteq \mathcal{P}g \cdot k$, there exists $k': X \rightarrow \mathcal{P}Y$ that for every $x \in X$, $k'(x) =$
 574 $k(x) \setminus \{y \mid g(y) \notin h(x)\}$. We show that for every $x \in X$, we have $\mathcal{P}g \cdot k'(x) = h(x)$.

575 Assuming $z \in \mathcal{P}g(k'(x))$, then there exist $y' \in k(x) \setminus \{y \mid g(y) \notin h(x)\}$ that
 576 $g(y') = z$, so $y' \in k(x)$, $y' \notin \{y \mid g(y) \notin h(x)\}$. $y' \notin \{y \mid g(y) \notin h(x)\}$ means that
 577 $g(y') \in h(x)$, thus $\mathcal{P}g \cdot k' \subseteq h$.

Assuming $z \in h(x)$, since $h \subseteq \mathcal{P}g \cdot k$, then $z \in \mathcal{P}g \cdot k(x)$. So, there exists y'
 such that $y' \in k(x)$ and $g(y') = z$. Since $g(y') \in h(x)$ then $y' \notin \{y \mid g(y) \notin h(x)\}$
 meaning that $y' \in k'(x)$ that means $z \in \mathcal{P}g(k'(x))$, thus $h \subseteq \mathcal{P}g \cdot k'$. \square

578 **Example 4.48.** In the category of sets, subset over the powerset functor is NOT
 579 an example of a cogood order! For some set X we take $h: X \rightarrow \mathcal{P}\mathbb{Z}$, for every
 580 $x \in X$, $h(x) = \mathbb{Z}$, $g: \mathbb{Z} \rightarrow \mathbb{Z}$, and for every $z \in \mathbb{Z}$,

$$g(z) = \begin{cases} 0 & z \in \mathbb{Z}^+ \\ 1 & \text{otherwise} \end{cases}$$

581 then

$$\mathcal{P}g(A) = \begin{cases} \emptyset & A = \emptyset \\ \{0\} & A \subseteq \mathbb{Z}^+, A \neq \emptyset \\ \{1\} & A \subseteq \mathbb{Z}^- \cup \{0\}, A \neq \emptyset \\ \{0, 1\} & \text{otherwise} \end{cases}$$

582 then no matter what $k: X \rightarrow \mathcal{P}\mathbb{Z}$ is there will be no $k': X \rightarrow \mathcal{P}\mathbb{Z}$, for every
 583 $x \in X$, $\mathcal{P}g(k'(x)) = h(x)$, as $\mathcal{P}g(k'(x)) \subseteq \{0, 1\}$, while $h(x) = \mathbb{Z}$, so $\mathcal{P}g(k'(x)) \subset$
 584 $h(x)$.

585 The previous example suggests that cogoodness is a strong condition. Perhaps it
 586 can be limited to be meaningful. Maybe the morphism g can be limited to legs
 587 of a span $(\pi_1: A \rightarrow X, \pi_2: A \rightarrow Y)$, where for a relation r , $r = \pi_2 \cdot \pi_1^{\text{op}}$.

588 **Proposition 4.49.** *For a natural order structure \sqsubseteq on a set-functor F , if for*
 589 *every $f \in \text{Hom}(X, FY)$ we have $Ff \cdot \sqsubseteq = \sqsubseteq \cdot Ff$, then \sqsubseteq is cogood.*

590 *Proof. TODO: Investigate if it's true! If it is, is naturality*
 591 *necessary? It is really weird! If having a good order structure*
 592 *forces every witness for AM-simulation to have the left cell of*
 593 *the lax diagram as equality, and subset on powerset functor is a*
 594 *good order structure, then what is the counter-example number 4*
 595 *that you have?!*

596 **Proposition 4.50.** *Assuming that \mathbf{R} is a difunctionally functorial relator, then*
 597 *the symmetrization of the relator that takes $r: X \rightarrowtail Y$ to $\mathbf{R}r \cdot \sqsubseteq$ is a sound*
 598 *relator.*

599 *Proof. TODO: Finish.*

600 **Proposition 4.51.** *Assuming that \mathbf{R} is a relator over $F: \mathbf{Set} \rightarrow \mathbf{Set}$, and \sqsubseteq_X*
 601 *and \sqsubseteq_Y are posets over FX and FY respectively, then the symmetrization of*
 602 *the relator that takes $r: X \rightarrowtail Y$ to $\sqsubseteq_X; \mathbf{R}r; \sqsubseteq_Y$ is a Barr relator.*

603 *Proof. TODO: Finish.*